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Exercise Boundary of the American Put Near Maturity in an Exponential Lévy Model

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Abstract We study the behavior of the critical price of an American put option near maturity in the exponential Lévy model. In particular, we prove that, in situations where the limit of the critical price is equal to the strike price, the rate of convergence to the limit is linear if and only if the underlying Lévy process has finite variation. In the case of infinite variation, a variety of rates of convergence can be observed: we prove that, when the negative part of the Lévy measure exhibits an α-stable density near the origin, with $1 < \alpha < 2$, the convergence rate is ruled by $\theta^{1/\alpha}|\ln\theta|^{1-\frac{1}{\alpha}}$, where θ is time until maturity.

Keywords American put · free boundary · optimal stopping · variational inequality

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1 Introduction

The behavior of the exercise boundary of the American put near maturity is well understood in the Black-Scholes model. In particular, Barles-Burdeau-Romano-Samsoen [1] (see also [8]) showed that, in the absence of dividends, the distance between the strike price K and the critical price at time t, which

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we denote by $b^{BS}(t)$ satisfies

$$\lim_{t\to T} \frac{K - b^{BS}(t)}{\sigma K \sqrt{(T-t)|\ln(T-t)|}} = 1, \tag{1.1}$$

where T is the maturity, and σ is the volatility (see also [4] for higher order expansions).

The aim of this paper is to study the exercise boundary of the American put near maturity in exponential Lévy models. Note that Pham [16] proved that the estimate (1.1) holds in a jump diffusion model satisfying some conditions. We will first extend Pham's result to slightly more general situations and, then, we will concentrate on Lévy processes with no Brownian part. In a recent paper (see [10]), we characterized the limit of the critical price at maturity for general exponential Lévy models (see also Levendorskii [13] for earlier related results). In particular, we proved that, if the interest rate r and the dividend rate δ satisfy

$$r - \delta \geq \int (e^y - 1)_+ \, \nu(dy), \tag{1.2}$$

where ν is the Lévy measure of the underlying Lévy process, the limit of the critical price at maturity is equal to the strike price K. In the present paper, we limit our study to situations where the limit is equal to K. We refer to [12] for results when the limit is not K within the Black-Scholes framework.

The *early exercise premium* formula is crucial in our approach. This result was established by Carr-Jarrow-Myneni [3], Jacka [6] and Kim [7] in the Black-Scholes model, and by Pham [16] in the jump diffusion model. In this work, we extend it to an exponential Lévy model when the related Lévy process is of type B or C (see the definition p.4).

The paper is organized as follows. In Section 2, we recall some facts about the exponential Lévy model and the basic properties of the American put price in this model. In Section 3, we establish the early exercise premium representation, and we slightly extend Pham's result by showing that (1.1) remains true when the logarithm of the stock includes a diffusion component and a pure jump process with finite variation. In the fourth section, we prove that the convergence rate of the critical price is linear with respect to t when the logarithm of the stock price is a finite variation Lévy process (see Theorem 4.2). Section 5 deals with the case when the logarithm of the stock price is an infinite variation Lévy process. We show in this case that the convergence speed of the critical price is not linear (see Theorem 5.1). Finally, in Section 6, we study processes with a Lévy density which behaves asymptotically like an α-stable density in a negative neighborhood of the origin, with $1 < \alpha < 2$. In this case, the rate of convergence involves time to maturity to a power $1/\alpha$, together with a logarithmic term, with exponent $1 - \frac{1}{\alpha}$ (see Theorem 6.1). So, there is a logarithmic factor (as in the Black-Scholes case), in contrast with the finite variation setting where the behavior is purely linear.

Remark 1.1 Some of our results can probably be extended to models involving more general Markov processes, but this extension might be technically heavy. We refer to [14] for results on the behavior of option prices near maturity in a rather general context.

2 The model

2.1 Lévy processes

A real Lévy process $X = (X_t)_{t\geq 0}$ is a càdlàg[1] real valued stochastic process, starting from 0, with stationary and independent increments. The Lévy-Itô decomposition (see [17]) gives the following representation of X

$$X_t = \gamma t + \sigma B_t + Y_t, \quad t \geq 0, \tag{2.1}$$

where γ and σ are real constants, $(B_t)_{t\geq 0}$ is a standard Brownian motion, and the process Y can be written in terms of the jump measure J_X of X

$$Y_t = \int_0^t \int_{\{|x|>1\}} x J_X(ds, dx) + \int_0^t \int_{\{0<|x|\leq 1\}} x \tilde{J}_X(ds, dx), \quad t \geq 0. \tag{2.2}$$

Recall that J_X is a Poisson measure on $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$, with intensity ν, and $\tilde{J}_X(dt, dx) = J(dt, dx) - dt\nu(dx)$ is the compensated Poisson measure. The measure ν is a positive Radon measure on $\mathbb{R}\backslash\{0\}$, called the Lévy measure of X, and it satisfies

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty. \tag{2.3}$$

The Lévy-Itô decomposition entails that the distribution of X is uniquely determined by (σ^2, γ, ν), which is called the characteristic triplet of the process X. The characteristic function of X_t, for $t \geq 0$, is given by the Lévy-Khinchin representation (see [17])

$$\mathbb{E}[e^{iz.X_t}] = \exp[t\varphi(z)], \quad z \in \mathbb{R}, \tag{2.4}$$

with

$$\varphi(z) = -\frac{1}{2}\sigma^2 z^2 + i\gamma z + \int \left(e^{izx} - 1 - izx\mathbb{1}_{\{|x|\leq 1\}}\right)\nu(dx).$$

The Lévy process X is a Markov process and its infinitesimal generator is given by

$$\begin{aligned} Lf(x) = {} & \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + \gamma \frac{\partial f}{\partial x}(x) \\ & + \int \left(f(x+y) - f(x) - y\frac{\partial f}{\partial x}(x)\mathbb{1}_{\{|y|\leq 1\}}\right)\nu(dy), \end{aligned} \tag{2.5}$$

for every $f \in \mathcal{C}_b^2(\mathbb{R})$, where $\mathcal{C}_b^2(\mathbb{R})$ denotes the set of all bounded $\mathcal{C}^2$ functions with bounded derivatives.

We recall the following classification of Lévy processes (see [17]).

[1] The sample paths of X are right continuous with left limits.

Definition 2.1 Let X a real Lévy process with characteristic triplet (σ^2, γ, ν). We say that X is of

- **type** A, if $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$;
- **type** B, if $\sigma = 0$, $\nu(\mathbb{R}) = \infty$ and $\int_{\{|x|\leq 1\}} |x|\nu(\mathbb{R}) < \infty$ (infinite activity and finite variation);
- **type** C, If $\sigma > 0$ or $\int_{|x|\leq 1} |x|\nu(\mathbb{R}) = \infty$ (infinite variation).

We complete this section by recalling the so-called *compensation formula* (see [2], preliminary chapter). We denote by $\Delta X_t = X_t - X_{t^-}$ the jump of the process X at time t.

Proposition 2.2 *Let X be a real Lévy prosess and $\Phi : (t, \omega, x) \mapsto \Phi_t^x(\omega)$ a measurable nonnegative function on $\mathbb{R}^+ \times \Omega \times \mathbb{R}$, equipped with the σ-algebra $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$, where $\mathcal{P}$ is the predictable σ-algebra on $\mathbb{R}^+ \times \Omega$, and $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. We have,*

$$\mathbb{E}\left(\sum_{0\leq s<\infty} \mathbb{1}_{\{\Delta X_s \neq 0\}} \Phi_s^{\Delta X_s}\right) = \mathbb{E}\left[\int_0^\infty ds \int \nu(dy) \Phi_s^y\right]. \tag{2.6}$$

Remark 2.3 The equality (2.6) remains true if the non-negativity assumption on Φ_t^x is replaced by the condition

$$\mathbb{E}\left[\int_0^\infty ds \int \nu(dy) |\Phi_s^y|\right] < \infty.$$

2.2 The exponential Lévy model

In the exponential Lévy model, the price process $(S_t)_{t\in[0,T]}$ of the risky asset is given by

$$S_t = S_0 e^{(r-\delta)t + X_t}, \tag{2.7}$$

where the interest rate r, the dividend rate δ are nonnegative constants and $(X_t)_{t\in[0,T]}$ is a real Lévy process with characteristic triplet (σ^2, γ, ν). We include r and δ in (2.7) for ease of notation.

Under the pricing measure $\mathbb{P}$, the discounted, dividend adjusted stock price $(e^{-(r-\delta)t} S_t)_{t\in[0,T]}$ is a martingale, which is equivalent (see, for instance, [5]), to the two conditions

$$\int_{|x|\geq 1} e^x \nu(dx) < \infty \quad \text{and} \quad \frac{\sigma^2}{2} + \gamma + \int \left(e^x - 1 - x\mathbb{1}_{|x|\leq 1}\right) \nu(dx) = 0. \tag{2.8}$$

We suppose that these conditions are satisfied in the sequel. We deduce from (2.8) that the infinitesimal generator defined in (2.5) can be written as

$$Lf(x) = \frac{\sigma^2}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\right)(x) + \int \left(f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}(x)\right) \nu(dy). \tag{2.9}$$

The stock price $(S_t)_{t\in[0,T]}$ is also a Markov process and $S_t = S_0 e^{\tilde{X}_t}$, where $\tilde{X}$ is a Lévy process with characteristic triplet $(\sigma^2, r-\delta+\gamma, \nu)$. We denote by $\tilde{L}$ the infinitesimal generator of $\tilde{X}$. So, from (2.9), we have

$$\tilde{L}f(x) = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial x}(x) + \tilde{\mathcal{B}}f(x), \tag{2.10}$$

where

$$\tilde{\mathcal{B}}f(x) = \int \nu(dy)\left(f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}(x)\right).$$

2.3 The American put price

In this model, the value at time t of an American put with maturity T and strike price K is given by

$$P_t = ess \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}(e^{-r\tau}\psi(S_\tau) \mid \mathcal{F}_t),$$

where $\psi(x) = (K-x)_+$ and $\mathcal{T}_{t,T}$ denotes the set of stopping times satisfying $t \le \tau \le T$. The filtration $(\mathcal{F}_t)_{t\ge 0}$ is the usual augmentation of the natural filtration of X. It can be proved (see, for instance, [15]) that

$$P_t = P(t, S_t),$$

where,

$$P(t,x) = \sup_{\tau\in\mathcal{T}_{0,T-t}} \mathbb{E}(e^{-r\tau}\psi(S_\tau^x)), \tag{2.11}$$

with $S_t^x = xe^{(r-\delta)t+X_t}$. The following proposition follows easily from (2.11).

Proposition 2.4 *For $t \in [0,T]$, the function $x \mapsto P(t,x)$ is non-increasing and convex on $[0,+\infty)$.*

For $x \in [0,+\infty)$, the function $t \mapsto P(t,x)$ is continuous and nondecreasing on $[0,T]$.

Note that we also have $P(t,x) \ge P_e(t,x)$, where P_e denotes the European put price, defined by

$$P_e(t,x) = \mathbb{E}(e^{-r(T-t)}\psi(S_{T-t}^x)), \quad (t,x) \in [0,T]\times\mathbb{R}^+.$$

We proved in [10] that the American put price satisfies a variational inequality in the sense of distributions. It is more convenient to state this variational inequality after a logarithmic change of variable. Define

$$\tilde{P}(t,x) = P(t,e^x), \quad (t,x) \in [0,T]\times\mathbb{R}. \tag{2.12}$$

We have

$$\tilde{P}(t,x) = \sup_{\tau\in\mathcal{T}_{0,T-t}} \mathbb{E}(e^{-r\tau}\tilde{\psi}(x+\tilde{X}_\tau)),$$

where $\tilde{\psi}(x) = \psi(e^x) = (K-e^x)_+$.

Theorem 2.5 *(see [10]) The distribution $(\partial_t + \tilde{L} - r)\tilde{P}$ is a nonpositive measure on $(0,T)\times\mathbb{R}$, and, on the open set $\tilde{C}$ we have $(\partial_t + \tilde{L} - r)\tilde{P} = 0$, where $\tilde{C}$ is called the continuation region, defined by $\tilde{C} = \{(t,x) \in (0,T)\times\mathbb{R} \mid \tilde{P}(t,x) > \tilde{\psi}(x)\}$.*

2.4 The free boundary

Throughout this paper, we will assume that at least one of the following conditions is satisfied:

$$\sigma \neq 0, \quad \nu((-\infty,0)) > 0 \quad \text{or} \quad \int_{(0,+\infty)} (x \wedge 1)\nu(dx) = +\infty. \tag{2.13}$$

We then have $\mathbb{P}(X_t < A) > 0$, for all $t > 0$ and $A \in \mathbb{R}$, so that $P_e(t,x) > 0$ for every $(t,x) \in [0,T) \times \mathbb{R}_+$. We will also assume that $r > 0$. The *critical price* or *American critical price* at time $t \in [0,T)$ is defined by

$$b(t) = \inf\{x \geq 0 \mid P(t,x) > \psi(x)\}.$$

Note that, since $t \mapsto P(t,x)$ is nonincreasing, the function $t \mapsto b(t)$ is nondecreasing. It follows from (2.13) that $b(t) \in [0,K)$. We obviously have $P(t,x) = \psi(x)$ for $x \in [0, b(t))$, and also for $x = b(t)$, due to the continuity of P and ψ. We also deduce from the convexity of $x \mapsto P(t,x)$ that

$$\forall t \in [0,T), \quad \forall x > b(t), \quad P(t,x) > \psi(x).$$

In other words the continuation region $\tilde{C}$ can be written as

$$\tilde{C} = \{(t,x) \in [0,T) \times [0,+\infty) \mid x > \tilde{b}(t)\},$$

where $\tilde{b}(t) = \ln(b(t))$. The graph of b is called the *exercise boundary* or *free boundary.*

It was proved in [10] that the function b is continuous on $[0,T)$, and that $b(t) > 0$. We also recall the following result, characterising the limit of the critical price near maturity (see [10] Theorem 4.4).

Theorem 2.6 *Denote*

$$d^+ = r - \delta - \int (e^x - 1)_+ \nu(dx).$$

If $d^+ \geq 0$, we have $\lim_{t\to T} b(t) = K$.

If $d^+ < 0$, we have $\lim_{t\to T} b(t) = \xi$, where ξ is the unique real number in the interval $(0,K)$ such that $\varphi_0(\xi) = rK$, where φ_0 is the function defined by $\varphi_0(x) = \delta x + \int (xe^y - K)_+ \nu(dy)$, $x \in (0,K)$.

3 The early exercise premium formula

The early exercise premium is the difference $P - P_e$ between the American and the European put prices. It can be expressed with the help of the exercise boundary. This expression can be deduced from the following Proposition, which characterises the distribution $(\partial_t + \tilde{L} - r)\tilde{P}$ as a bounded measurable function, with a simple expression involving the exercise boundary.

Proposition 3.1 *The distribution* $(\partial_t + \tilde{L} - r)\tilde{P}$ *is given by*

$$(\partial_t + \tilde{L} - r)\tilde{P}(t,x) = h(t,x), \quad dtdx\text{-a.e. on } (0,T)\times\mathbb{R}^+, \tag{3.1}$$

where h is the function defined by

$$h(t,x) = \left[\delta e^x - rK + \int_{\{y>0\}} \left(\tilde{P}(t,x+y) - (K - e^{x+y})\right)\nu(dy)\right]\mathbb{1}_{\{x<\tilde{b}(t)\}}, \tag{3.2}$$

with $\tilde{b}(t) = \ln b(t)$.

Proof We know from Theorem 2.5 that, on the open set

$$\tilde{C} = \{(t,x)\in(0,T)\times\mathbb{R} \mid x > \tilde{b}(t)\},$$

we have $(\partial_t + \tilde{L} - r)\tilde{P}(t,x) = 0$. On the other hand, on the open set

$$\tilde{E} = \{(t,x)\in(0,T)\times\mathbb{R}^+ \mid \quad x < \tilde{b}(t)\},$$

we have $\tilde{P} = \tilde{\psi}$, so that, using (2.10) and $\tilde{\psi}(x) = K - e^x$, we have

$$\begin{aligned}
(\partial_t + \tilde{L} - r)\tilde{P}(t,x) &= \tilde{L}\tilde{P}(t,x) - r(K - e^x) \\
&= \delta e^x - rK + \tilde{\mathcal{B}}\tilde{P}(t,x) \\
&= \delta e^x - rK \\
&\quad + \int \nu(dy)\left(\tilde{P}(t,x+y) - \tilde{\psi}(x) - (e^y - 1)\tilde{\psi}'(x)\right) \\
&= \delta e^x - rK \\
&\quad + \int \nu(dy)\left(\tilde{P}(t,x+y) - (K - e^x) - (e^y - 1)(-e^x)\right) \\
&= \delta e^x - rK + \int \nu(dy)\left(\tilde{P}(t,x+y) + e^{x+y} - K\right).
\end{aligned}$$

At this point, we clearly have $(\partial_t + \tilde{L} - r)\tilde{P} = h$ on the open sets $\tilde{C}$ and $\tilde{E}$. Now, if $\sigma > 0$ and $\nu(\mathbb{R}) < \infty$, we know (cf. [18]) that the partial derivatives are locally bounded functions, so that the distribution $(\partial_t + \tilde{L} - r)\tilde{P} = h$ is in fact a locally bounded function, and, since the complement of $\tilde{C}\cup\tilde{E}$ is Lebesgue-negligible, we deduce (3.1). Now, observe that $h(t,x) \geq -rK$, so that we have $-rK \leq (\partial_t + \tilde{L} - r)\tilde{P} \leq 0$, at least if $\sigma > 0$ and $\nu(\mathbb{R}) < \infty$. On the other hand, in the general case, we can approximate (in law) the Lévy process X by a sequence of processes X^n with finite Lévy measures ν_n and positive Brownian variance parameters σ_n^2, in such a way that the American put prices P^n converge simply to P. We then have convergence of $(\partial_t+\tilde{L}-r)\tilde{P}_n$ to $(\partial_t+\tilde{L}-r)\tilde{P}$ in the sense of distributions, so that the double inequality $-rK \leq (\partial_t+\tilde{L}-r)\tilde{P} \leq 0$ is preserved in the limit. And we can conclude as in the special case that (3.1) is true.

The early exercise premium formula is given by the following theorem.

Theorem 3.2 *The American put price P, related to a Lévy process X of type B or C, has the following representation*

$$P(t,x) = P_e(t,x) + e(t,x),$$

where e is the early exercise premium*, defined by*

$$e(t,x) = \mathbb{E}\left(\int_0^{T-t} k(t+s, S_s^x)e^{-rs}ds\right),$$

and the function k is given by

$$k(t,x) = \left[rK - \delta x - \int_{\{y>0\}} \left(P(t,xe^y) - (K - xe^y)\right)\nu(dy)\right] \mathbb{1}_{\{x<b(t)\}}, \tag{3.3}$$

for every $(t,x) \in [0,T) \times \mathbb{R}^+$.

Proof We first extend the definition of $\tilde{P}$ by setting

$$\tilde{P}(t,x) = 0, \quad \text{for} \quad t \notin [0,T], \quad x \in \mathbb{R}.$$

Next, we regularize $\tilde{P}$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative C^∞ functions on $\mathbb{R}^2$, such that, for every $n \in \mathbb{N}$, $supp(\rho_n) \subset (-1/n, 1/n) \times (-1/n, 1/n)$ and $\int_{\mathbb{R}^2} \rho_n = 1$. Define

$$\tilde{P}_n(t,x) = (\tilde{P} * \rho_n)(t,x) = \int_{\mathbb{R}^2} \tilde{P}(t-v, x-y)\rho_n(v,y)dvdy, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Note that, for each n, the function $\tilde{P}_n$ is C^∞, with bounded derivatives, and that we have

$$\forall (t,x) \in (0,T)\times\mathbb{R}, \quad 0 \leq \tilde{P}_n(t,x) \leq K \quad \text{and} \quad \lim_{n\to\infty} \tilde{P}_n(t,x) = P(t,x). \tag{3.4}$$

Now, fix t in the open interval $(0,T)$, and let

$$f_n(s,y) = \tilde{P}_n(t+s, y), \quad (s,y) \in \mathbb{R} \times \mathbb{R}.$$

Since f_n is smooth with bounded derivatives, we have for any time t_1, with $0 < t_1 < T-t$ and any $x \in \mathbb{R}$,

$$\mathbb{E}\left(e^{-rt_1} f_n(t_1, x + \tilde{X}_{t_1})\right) = f_n(0,x) + \mathbb{E}\left[\int_0^{t_1} e^{-rs}\left(\partial_s + \tilde{L} - r\right) f_n(s, x+\tilde{X}_s)ds\right]. \tag{3.5}$$

Recall that $(\tilde{X}_t)_{t\in[0,T]}$ is defined by $\tilde{X}_t = (r-\delta)t + X_t$, and that $\tilde{L}$ is the infinitesimal generator of $\tilde{X}$.

We have, using Proposition 3.1 and the equality $\tilde{L}(\rho_n * \tilde{P}) = \rho_n * \tilde{L}\tilde{P}$ (see [10]),

$$\left(\partial_s + \tilde{L} - r\right) f_n = \rho_n * \left(\partial_s + \tilde{L} - r\right) \tilde{P}_n(t+\cdot,\cdot) = \rho_n * h(t+\cdot,\cdot).$$

Note that $-rK \le h \le 0$, and it follows from (3.2) that h is continuous on the set $\{(s,y) \mid 0 < s < T \text{ and } y \ne \tilde{b}(s)\}$ (note that, for the continuity of the integral, a domination condition can be deduced from the fact that $x \mapsto P(t,x) - (K-x)$ is non decreasing, as follows from the convexity of $P(t,.)$). Now, since $\tilde{X}$ is a Lévy process of type B or C, we have, for every $s \in (0, T-t)$ (see [17]),

$$\mathbb{P}\left(x + \tilde{X}_s = \tilde{b}(t+s)\right) = 0,$$

so that, by dominated convergence,

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{t_1} e^{-rs}\left(\partial_s + \tilde{L} - r\right) f_n(s, x+\tilde{X}_s)ds\right] = \mathbb{E}\left[\int_0^{t_1} e^{-rs} h(t+s, x+\tilde{X}_s)ds\right].$$

On the other hand, using (3.4), we have $\lim_{n\to\infty} f_n(0,x) = \tilde{P}(t,x)$ and, by passing to the limit in (3.5),

$$\mathbb{E}\left(e^{-rt_1}\tilde{P}(t+t_1, x+\tilde{X}_{t_1})\right) = \tilde{P}(t,x) + \mathbb{E}\left[\int_0^{t_1} e^{-rs} h(t+s, x+\tilde{X}_s)ds\right].$$

Now, take the limit as $t_1 \to T-t$, and use the continuity of $\tilde{P}$ on $[0,T]\times\mathbb{R}$ to derive

$$\mathbb{E}\left(e^{-r(T-t)}\tilde{P}(T, x+\tilde{X}_{T-t})\right) = \tilde{P}(t,x) + \mathbb{E}\left[\int_0^{T-t} e^{-rs} h(t+s, x+\tilde{X}_s)ds\right].$$

We have $P(t,x) = \tilde{P}(t, \ln x)$ and

$$P_e(t,x) = \mathbb{E}\left(e^{-r(T-t)}\left(K - xe^{\tilde{X}_{T-t}}\right)_+\right) = \mathbb{E}\left(e^{-r(T-t)}\tilde{P}(T, \ln x + \tilde{X}_{T-t})\right),$$

so that

$$P(t,x) = P_e(t,x) - \mathbb{E}\left[\int_0^{T-t} e^{-rs} h(t+s, \ln x + \tilde{X}_s)ds\right],$$

and the early exercise premium formula follows, using the equality $k(t,x) = -h(t, \ln x)$.

Remark 3.3 It follows from Proposition 3.1 that $h \geq -rK\mathbb{1}_{x<\tilde{b}(t)}$, so that, for $t \in (0,T)$ and $s \in (0, T-t)$,

$$\liminf_{n\to\infty} \rho_n * h(t+s, x+\tilde{X}_s) \geq -rK\mathbb{1}_{\{x+\tilde{X}_s \leq \tilde{b}(t+s)\}}.$$

Using this inequality, we deduce from the proof of Theorem 3.2 that (even if X is not of type B or C),we have

$$0 \leq P(t,x) - P_e(t,x) \leq rK\mathbb{E}\left(\int_0^{T-t} \mathbb{1}_{\{S_s^x \leq b(t+s)\}} ds\right).$$

The following consequence of Theorem 3.2 will be useful in Section 6.

Corollary 3.4 *For every $t \in [0,T)$, the function $x \mapsto P(t,x) - P_e(t,x)$ is nonincreasing on $\mathbb{R}^+$.*

Proof It suffices to show that the early exercise premium $e(t,x)$ in Theorem 3.2 is a nonincreasing function of x. This will clearly follow if we prove that $x \mapsto k(t,x)$ is nonincreasing, where k is the function defined in (3.3). Note that, due to the convexity of $P(t,.)$, the function $x \mapsto P(t,x) - (K-x)$ is nondecreasing, so that

$$x \mapsto rK - \delta x - \int_{\{y>0\}} \left(P(t, xe^y) - (K - xe^y)\right)\nu(dy),$$

is nonincreasing.

Corollary 3.4 was proved for jump-diffusion models by PDE arguments in [16], and was one of the ingredients for establishing the rate of convergence of the critical price to its limit. In fact, by following Pham's proof, we can extend his result in the following form (details can be found in [9]).

Theorem 3.5 *Assume $\sigma > 0$ and $\int_{\{|x|\leq 1\}} |x|\nu(dx) < \infty$. If $d^+ > 0$, we have*

$$\lim_{t\to T} \frac{b(t) - K}{\sigma K\sqrt{(T-t)|\ln(T-t)|}} = 1.$$

Remark 3.6 It is quite likely that the result of Theorem 3.5 is also true in the case of a jump part with infinite variation, but we have not been able to prove it. In fact, one of the arguments needed in Pham's proof involves an estimate on the difference of the American put prices in the Black-Scholes model and the jump diffusion model. This difference is clearly $O(\theta)$, where θ is time until maturity, in the case of finite variation, but, in the case of infinite variation, it might be different.

4 The critical price near maturity in a finite variation Lévy model

Throughout this section, we suppose that X is a Lévy process with finite variation, or, equivalently,

$$\sigma = 0 \quad \text{and} \quad \int_{|x|\leq 1} |x|\nu(dx) < \infty.$$

The decomposition (2.1) can then be written as follows

$$X_t = \gamma_0 t + \sum_{0<s\leq t} \Delta X_s, \quad t \geq 0, \tag{4.1}$$

where $\gamma_0 = \gamma - \int_{|x|\leq 1} x\nu(dx)$. Note that, due to the martingale condition (2.8), we have

$$\gamma_0 = -\int (e^y - 1)\nu(dy). \tag{4.2}$$

This section is divided into two parts. In the first part, we introduce what we call the *European* critical price, namely the stock price value for which the American put price is equal to its intrinsic value, and we characterise its behavior near maturity. In the second part, we analyze the difference between the *European* and *American* critical prices and deduce the behavior of the American critical price.

4.1 The European critical price

For each t in the interval $[0, T)$, we define the *European critical price* at time $t \in [0, T)$ by

$$b_e(t) = \inf\{x \in \mathbb{R}_+ \mid P_e(t, x) > \varphi(x)\}.$$

Note that, since $P_e(t, K) > 0$ and $P_e(t, 0) = Ke^{-r(T-t)}$, we have $0 < b_e(t) < K$. Using the convexity of $P_e(t, \cdot)$, one can see that $b_e(t)$ is the only real number in the interval $(0, K)$ satisfying the equality $P_e(t, b_e(t)) = K - b_e(t)$. Recall that $P \geq P_e$, so that $b \leq b_e \leq K$, and it follows from Theorem 2.6 that, if $d^+ \geq 0$, we have $\lim_{t\to T} b_e(t) = \lim_{t\to T} b(t) = K$. The following result characterises the rate of convergence of $b_e(t)$ to K.

Theorem 4.1 *If $d^+ > 0$, we have*

$$\lim_{t\to T} \left[\frac{1}{T-t}\left(\frac{K}{b_e(t)} - 1\right)\right] = \int (e^y - 1)_- \nu(dy).$$

Proof Starting from the equality $P_e(t, b_e(t)) = K - b_e(t)$, we have, with the notation $\theta = T - t$,

$$\begin{aligned}
K - b_e(t) &= \mathbb{E}(e^{-r\theta}(K - b_e(t)e^{(r-\delta)\theta + X_\theta})_+) \\
&= e^{-r\theta}K - b_e(t)\mathbb{E}e^{-\delta\theta + X_\theta} + \mathbb{E}(e^{-r\theta}(K - b_e(t)e^{(r-\delta)\theta+X_\theta})_-) \\
&= e^{-r\theta}K - b_e(t)e^{-\delta\theta} + \mathbb{E}(e^{-r\theta}(b_e(t)e^{(r-\delta)\theta+X_\theta} - K)_+),
\end{aligned}$$

Dividing both sides by $b_e(t)$, we get

$$\frac{K}{b_e(t)}(1-e^{-r\theta})+e^{-\delta\theta}-1=\mathbb{E}\left[e^{-r\theta}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+\right].$$

Note that, since $\lim_{t\to T} b_e(t)=K$,

$$\frac{K}{b_e(t)}(1-e^{-r\theta})+e^{-\delta\theta}-1=(r-\delta)\theta+o(\theta).$$

Therefore, using the decomposition (4.1),

$$\begin{aligned}(r-\delta)\theta &= \mathbb{E}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+ + o(\theta)\\ &= \mathbb{E}\left(e^{(r-\delta+\gamma_0)\theta+Z_\theta}-\frac{K}{b_e(t)}\right)_+ + o(\theta),\end{aligned} \tag{4.3}$$

with the notation

$$Z_t=\sum_{0<s\le t}\Delta X_s,\quad t\ge 0.$$

We have

$$\begin{aligned}\mathbb{E}\left(e^{(r-\delta+\gamma_0)\theta+Z_\theta}-\frac{K}{b_e(t)}\right)_+ &= \mathbb{E}\left(e^{Z_\theta}[1+(r-\delta+\gamma_0)\theta]-\frac{K}{b_e(t)}\right)_+ + o(\theta)\\ &= \mathbb{E}\left(e^{Z_\theta}+(r-\delta+\gamma_0)\theta-\frac{K}{b_e(t)}\right)_+ + o(\theta),\end{aligned}$$

where the last equality follows from the fact that $\lim_{\theta\to 0}\mathbb{E}|e^{Z_\theta}-1|=0$. Going back to (4.3), we deduce

$$(r-\delta)\theta=\mathbb{E}(f_\theta(Z_\theta))+o(\theta), \tag{4.4}$$

where the function f_θ is defined by

$$f_\theta(x)=(e^x-1-\tilde{\zeta}(\theta))_+,\quad x\in\mathbb{R},$$

with

$$\tilde{\zeta}(\theta)=\frac{K}{b_e(t)}-1-(r-\delta+\gamma_0)\theta.$$

Since the process Z is the sum of its jumps, we have, using the compensation formula (see Proposition 2.2),

$$\begin{aligned}\mathbb{E}(f_\theta(Z_\theta)) &= f_\theta(0)+\mathbb{E}\left(\sum_{0<s\le\theta}[f_\theta(Z_s)-f_\theta(Z_{s^-})]\right)\\ &= f_\theta(0)+\mathbb{E}\left(\int_0^\theta ds\int(f_\theta(Z_s+y)-f_\theta(Z_s))\nu(dy)\right)\\ &= ((r-\delta+\gamma_0)\theta-\zeta(\theta))_+ + \int_0^\theta ds\int\nu(dy)\mathbb{E}\left(f_\theta(Z_s+y)-f_\theta(Z_s)\right),\end{aligned}$$

with

$$\zeta(\theta) = \tilde{\zeta}(\theta) + (r - \delta + \gamma_0)\theta = \frac{K}{b_e(t)} - 1.$$

Note that, since $\lim_{\theta \downarrow 0} \tilde{\zeta}(\theta) = 0$, we have, for any fixed $y \in \mathbb{R}$,

$$\begin{aligned}
\lim_{\theta \downarrow 0} \frac{1}{\theta} \int_0^\theta ds \mathbb{E}\left(f_\theta(Z_s + y)\right) &= \lim_{\theta \downarrow 0} \frac{1}{\theta} \int_0^\theta ds \mathbb{E}\left(e^{Z_s + y} - 1 - \tilde{\zeta}(\theta)\right)_+ \\
&= \lim_{\theta \downarrow 0} \frac{1}{\theta} \int_0^\theta ds \mathbb{E}\left(e^{Z_s + y} - 1\right)_+ \\
&= \left(e^y - 1\right)_+,
\end{aligned}$$

where the last equality follows from the fact that $\lim_{s \to 0} e^{Z_s} = 1$ in L_1.

On the other hand, we have

$$\begin{aligned}
\frac{1}{\theta} \int_0^\theta ds \mathbb{E}\left(f_\theta(Z_s + y) - f_\theta(Z_s)\right) &\leq \frac{1}{\theta} \int_0^\theta ds \mathbb{E}\left(e^{Z_s} \left|e^y - 1\right|\right) \\
&= \frac{1}{\theta} \int_0^\theta ds e^{-\gamma_0 s} \left|e^y - 1\right| \leq \frac{e^{|\gamma_0|\theta} - 1}{|\gamma_0|\theta} \left|e^y - 1\right|.
\end{aligned}$$

Since $\sup_{0<\theta<1} \dfrac{e^{|\gamma_0|\theta} - 1}{|\gamma_0|\theta} < \infty$ and $\int \left|e^y - 1\right| \nu(dy) < \infty$, we deduce, by dominated convergence, that

$$\lim_{\theta \downarrow 0} \frac{1}{\theta} \int_0^\theta ds \int \nu(dy) \mathbb{E}\left(f_\theta(Z_s + y) - f_\theta(Z_s)\right) = \int \left(e^y - 1\right)_+ \nu(dy).$$

We can now rewrite (4.4) as

$$(r - \delta)\theta = ((r - \delta + \gamma_0)\theta - \zeta(\theta))_+ + \theta \int (e^y - 1)_+ \nu(dy) + o(\theta),$$

so that

$$d^+\theta = \left(r - \delta - \int (e^y - 1)_+ \nu(dy)\right)\theta = ((r - \delta + \gamma_0)\theta - \zeta(\theta))_+ + o(\theta).$$

Since $d^+ > 0$, we must have $(r - \delta + \gamma_0)\theta - \zeta(\theta) > 0$ for θ close to 0. Hence

$$\lim_{\theta \downarrow 0} \frac{\zeta(\theta)}{\theta} = \gamma_0 + \int (e^y - 1)_+ \nu(dy) = \int (e^y - 1)_- \nu(dy),$$

where the last equality follows from (4.2).

4.2 The behavior of the critical price

We are now in a position to prove the main result of this section.

Theorem 4.2 *If $d^+ > 0$, we have*

$$\lim_{t\to T} \frac{1}{T-t}\left(\frac{K}{b(t)} - 1\right) = \int (e^y - 1)_- \nu(dy).$$

Proof In view of Theorem 4.1, it suffices to prove that

$$\lim_{t\to T} \frac{b_e(t) - b(t)}{(T-t)} = 0.$$

Recall that $b_e \geq b$ and, from Remark 3.3, we have, for $(t,x) \in [0,T) \times \mathbb{R}^+$,

$$0 \leq P(t,x) - P_e(t,x) \leq rK\mathbb{E}\left(\int_0^{T-t} \mathbb{1}_{\{S_s^x \leq b(t+s)\}} ds\right). \tag{4.5}$$

From the equality $P_e(t, b_e(t)) = K - b_e(t)$ and the convexity of $P(t,\cdot)$, we deduce

$$\begin{aligned} P(t,b_e(t)) - P_e(t,b_e(t)) &= P(t,b_e(t)) - (K - b_e(t)) \\ &\geq P(t,b(t)) + (b_e(t) - b(t))\partial_x^+ P(t,b(t)) - (K - b_e(t)) \\ &= (b_e(t) - b(t))(\partial_x^+ P(t,b(t)) + 1). \end{aligned} \tag{4.6}$$

We now use the following lower bound for the jump of derivative of $P(t,\cdot)$ at $b(t)$ (see [11], Remark 4.1).

$$\partial_x^+ P(t,b(t)) + 1 \geq \frac{d^+}{d},$$

with $d = d^+ + \int (e^y - 1)_- \nu(dy)$. By combining (4.5) and (4.6), we get

$$0 \leq b_e(t) - b(t) \leq \frac{rKd}{d^+}\mathbb{E}\left(\int_0^{T-t} \mathbb{1}_{\{S_s^{b_e(t)} \leq b(t+s)\}} ds\right).$$

We now want to prove that

$$\lim_{t\to T} \frac{1}{T-t}\mathbb{E}\left(\int_0^{T-t} \mathbb{1}_{\{S_s^{b_e(t)} \leq b(t+s)\}} ds\right) = 0. \tag{4.7}$$

We first note that

$$\mathbb{E}\left(\int_0^{T-t} \mathbb{1}_{\{S_s^{b_e(t)} \leq b(t+s)\}} ds\right) = \int_0^{T-t} \mathbb{P}\left((r-\delta)s + X_s \leq \ln\left(\frac{b(t+s)}{b_e(t)}\right)\right) ds.$$

Using the notation $\theta = T-t$ and $\zeta(u)=\frac{K}{b_e(T-u)}$, for $u\in(0,T]$, we have

$$\begin{aligned}\ln\left(\frac{b(t+s)}{b_e(t)}\right) &\le \ln\left(\frac{b_e(t+s)}{b_e(t)}\right)\\ &\le \frac{b_e(t+s)}{b_e(t)}-1=\frac{\zeta(\theta)-\zeta(\theta-s)}{\zeta(\theta-s)}\le |\zeta(\theta)-\zeta(\theta-s)|,\end{aligned}$$

since $\zeta\ge 1$. Therefore,

$$\mathbb{E}\left(\int_0^\theta \mathbb{1}_{\{S_s^{b_e(t)}<b(t+s)\}}ds\right)\le \int_0^\theta \mathbb{P}\left((r-\delta)s+X_s\le |\zeta(\theta)-\zeta(\theta-s)|\right)ds. \quad (4.8)$$

It follows from Theorem 4.1 that

$$\lim_{u\to 0}\frac{\zeta(u)}{u}=\int (e^y-1)_-\nu(dy).$$

Therefore, given any $\varepsilon>0$, there exists $\eta_\varepsilon>0$ such that, for $u\in(0,\eta_\varepsilon]$,

$$-\varepsilon+\int (e^y-1)_-\nu(dy)\le \frac{\zeta(u)}{u}\le \varepsilon+\int(e^y-1)_-\nu(dy).$$

Take $\theta\in]0,\eta_\varepsilon]$ and $s\in]0,\theta]$. We have

$$\begin{aligned}\zeta(\theta)-\zeta(\theta-s)&\le \theta\left(\varepsilon+\int(e^y-1)_-\nu(dy)\right)-(\theta-s)\left(-\varepsilon+\int(e^y-1)_-\nu(dy)\right)\\ &= s\int(e^y-1)_-\nu(dy)+2\theta\varepsilon-s\varepsilon\\ &\le s\int(e^y-1)_-\nu(dy)+2\theta\varepsilon.\end{aligned}$$

Hence, using (4.1) and (4.2), we get, with the notation $Z_s=X_s-\gamma_0 s$,

$$\begin{aligned}\mathbb{P}\left((r-\delta)s+X_s\le|\zeta(\theta)-\zeta(\theta-s)|\right)&\le \mathbb{P}\left((r-\delta)s+X_s\le s\int(e^y-1)_-\nu(dy)+2\theta\varepsilon\right)\\ &=\mathbb{P}\left(Z_s\le -s\left(r-\delta+\gamma_0-\int(e^y-1)_-\nu(dy)\right)+2\theta\varepsilon\right)\\ &=\mathbb{P}\left(Z_s\le -sd^++2\theta\varepsilon\right). \quad (4.9)\end{aligned}$$

Now, take $\varepsilon<\frac{d^+}{4}$ and $\theta\le\eta_\varepsilon$. We deduce from (4.8) and (4.9) that

$$\begin{aligned}\mathbb{E}\left(\int_0^\theta \mathbb{1}_{\{S_s^{b_e(t)}\le b(t+s)\}}ds\right)&\le\int_0^\theta \mathbb{P}\left(Z_s\le -sd^++2\theta\varepsilon\right)ds\\ &=\int_0^{\frac{4\theta\varepsilon}{d^+}}\mathbb{P}\left(Z_s\le -sd^++2\theta\varepsilon\right)ds+\int_{\frac{4\theta\varepsilon}{d^+}}^\theta \mathbb{P}\left(Z_s\le -sd^++2\theta\varepsilon\right)ds\\ &\le \frac{4\theta\varepsilon}{d^+}+\int_{\frac{4\theta\varepsilon}{d^+}}^\theta \mathbb{P}\left(\frac{Z_s}{s}\le -d^++\frac{2\theta\varepsilon}{s}\right)ds\\ &\le \frac{4\theta\varepsilon}{d^+}+\int_{\frac{4\theta\varepsilon}{d^+}}^\theta \mathbb{P}\left(\frac{Z_s}{s}\le -\frac{d^+}{2}\right)ds\le\frac{4\theta\varepsilon}{d^+}+\int_0^\theta\mathbb{P}\left(\frac{Z_s}{s}\le-\frac{d^+}{2}\right)ds.\end{aligned}$$

Since the process Z has no drift part, we have $\lim_{s\to 0}\dfrac{Z_s}{s}=0$ a.s. (see [17], Section 47), so that

$$\lim_{s\to 0}\mathbb{P}\left(\frac{Z_s}{s}\leq -\frac{d^+}{2}\right)=0.$$

Hence

$$\limsup_{\theta\downarrow 0}\frac{1}{\theta}\mathbb{E}\left(\int_0^\theta \mathbb{1}_{\{S_s^{b_e(t)}\leq b(t+s)\}}ds\right)\leq\frac{4\varepsilon}{d^+}.$$

Since ε can be arbitrarily close to 0, (4.7) is proved.

5 The critical price near maturity in an infinite variation Lévy model

Throughout this section, we assume that X is an infinite variation Lévy process i.e.

$$\sigma\neq 0\quad\text{or}\quad\int_{|x|\leq 1}|x|\nu(dx)=\infty.$$

Our main result is that, in this case, the convergence of $b(t)$ to K cannot be linear.

Theorem 5.1 *Assume that X is Lévy process with infinite variation. If $d^+\geq 0$, we have*

$$\lim_{t\to T}\frac{1}{T-t}\left(\frac{K}{b(t)}-1\right)=\infty.$$

This result follows from the following Lemma, which will be proved later.

Lemma 5.2 *If X is a Lévy process with infinite variation, we have*

$$\lim_{t\to 0}\mathbb{E}\left(\frac{X_t}{t}\right)_+=\infty.$$

Proof of Theorem 5.1 We use the notation $\theta=T-t$. From the equality $P_e(t,b_e(t))=K-b_e(t)$, we derive, as in the proof of Theorem 4.1 (see (4.3)), that

$$(r-\delta)\theta=\mathbb{E}\left[\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+\right]+o(\theta)$$

Denote $\zeta(\theta)=\frac{K}{b_e(t)}-1$. Using the inequality $e^x\geq x+1$, we deduce

$$\begin{aligned}(r-\delta)\theta&=\mathbb{E}\left[\left(e^{(r-\delta)\theta+X_\theta}-1-\zeta(\theta)\right)_+\right]+o(\theta)\\&\geq\mathbb{E}\left(\left((r-\delta)\theta+X_\theta-\zeta(\theta)\right)_+\right)+o(\theta)\\&\geq\mathbb{E}\left[(r-\delta)\theta+X_\theta\right]_+-\zeta(\theta)+o(\theta)\\&=\mathbb{E}\left(\tilde{X}_\theta\right)_+-\zeta(\theta)+o(\theta),\end{aligned}$$

where $\tilde{X}_t = (r-\delta)t + X_t$. Therefore,

$$\liminf_{\theta\downarrow 0} \frac{\zeta(\theta)}{\theta} \geq \lim_{\theta\downarrow 0} \mathbb{E}\left(\frac{\tilde{X}_\theta}{\theta}\right)_+ - (r-\delta).$$

Since $\tilde{X}$ is a Lévy process with infinite variation, the Theorem follows from Lemma 5.2. $\diamond$

Proof of Lemma 5.2 Denote by (σ^2, γ, ν) the characteristic triplet of X. The Lévy-Itô decomposition of X can be written (see (2.1) and (2.2))

$$X_t = \gamma_t + \sigma B_t + \hat{X}_t + X_t^0, \quad t \geq 0,$$

with

$$\hat{X}_t = \int_0^t \int_{\{|x|>1\}} x J_X(ds, dx) \quad \text{and} \quad \tilde{X}_t^0 = \int_0^t \int_{\{0<|x|\leq 1\}} x \tilde{J}_X(ds, dx),$$

where J_X is the jump measure of X. Note that $\hat{X}$ is a compound Poisson process. We have

$$\begin{aligned}(X_t)_+ &= \left(\gamma_t + \sigma B_t + \tilde{X}_t^0 + \hat{X}_t\right) \mathbb{1}_{\left\{\gamma_t + \sigma B_t + \tilde{X}_t^0 + \hat{X}_t \geq 0\right\}} \\ &\geq \left(\gamma t + \sigma B_t + \tilde{X}_t^0\right) \mathbb{1}_{\left\{\gamma t + \sigma B_t + \tilde{X}_t^0 \geq 0\right\}} \mathbb{1}_{\left\{\hat{X}_t = 0\right\}}.\end{aligned}$$

Since B, $\hat{X}$ and $\tilde{X}^0$ are independent, we have

$$\begin{aligned}\mathbb{E}\left(\frac{X_t}{t}\right)_+ &\geq \mathbb{E}\left(\frac{\sigma B_t + \tilde{X}_t^0}{t} + \gamma\right)_+ \mathbb{P}(\hat{X}_t = 0) \\ &\geq \mathbb{E}\left(\frac{\sigma B_t + \tilde{X}_t^0}{t} + \gamma\right)_+ e^{-t\nu(\{|x|\geq 1\})},\end{aligned}$$

where the last inequality follows from the fact that the first jump time of the process $\hat{X}$ is exponentially distributed with parameter $\nu(\{|x| \geq 1\})$. Since $\left(\frac{\sigma B_t + \bar{X}_t^0}{t} + \gamma\right)_+ \geq \left(\frac{\sigma B_t + \bar{X}_t^0}{t}\right)_+ - |\gamma|$, it suffices to show that

$$\lim_{t\to 0} \mathbb{E}\left[\left(\frac{\sigma B_t + \tilde{X}_t^0}{t}\right)_+\right] = \infty. \tag{5.1}$$

For that, we discuss two cases.

We first assume that $\sigma \neq 0$. Recall that B and $\tilde{X}^0$ are independent and $\mathbb{E}(\tilde{X}_t^0) = 0$. By conditioning on B and using Jensen's inequality, we get

$$\begin{aligned}\mathbb{E}\left(\frac{\sigma B_t + \tilde{X}_t^0}{t}\right)_+ &= \mathbb{E}\left[\mathbb{E}\left(\left(\frac{\sigma B_t + \tilde{X}_t^0}{t}\right)_+ \mid B_t\right)\right] \\ &\geq \mathbb{E}\left[\left(\mathbb{E}\left(\frac{\sigma B_t + \tilde{X}_t^0}{t} \mid B_t\right)\right)_+\right] = \mathbb{E}\left[\left(\sigma\frac{B_t}{t}\right)_+\right] = |\sigma|\frac{1}{\sqrt{2\pi t}},\end{aligned}$$

so that (5.1) is proved.

Now, assume $\sigma = 0$. Since the process X has infinite variation, we must have $\int_{|y|\leq 1} |y|\nu(dy) = \infty$. Given $\varepsilon \in (0,1)$ and $t > 0$, introduce

$$\begin{aligned}\tilde{X}_t^\varepsilon &= \int_0^t \int_{\{\varepsilon \leq |x| \leq 1\}} x\tilde{J}_X(ds,dx) \\ &= X_t^\varepsilon - C_\varepsilon t,\end{aligned}$$

where

$$X_t^\varepsilon = \sum_{0\leq s\leq t} \Delta X_s \mathbb{1}_{\{\varepsilon \leq |\Delta X_s| \leq 1\}} \quad \text{and} \quad C_\varepsilon = \int_{\{\varepsilon\leq |y|\leq 1\}} y\nu(dy).$$

We have $\tilde{X}_t^0 = \tilde{X}_t^\varepsilon + (\tilde{X}_t^0 - \tilde{X}_t^\varepsilon)$, and the random variables $\tilde{X}_t^\varepsilon$ and $\tilde{X}_t^0 - \tilde{X}_t^\varepsilon$ are independent and centered. Therefore

$$\mathbb{E}\left[\left(\frac{\tilde{X}_t^0}{t}\right)_+\right] \geq \mathbb{E}\left[\left(\frac{\tilde{X}_t^\varepsilon}{t}\right)_+\right] = \mathbb{E}\left(\frac{X_t^\varepsilon - tC_\varepsilon}{t}\right)_+. \tag{5.2}$$

We have $\mathbb{E}\left(X_t^\varepsilon - tC_\varepsilon\right)_+ = \mathbb{E}g_t(X_t^\varepsilon)$, with $g_t(x) = (x - tC_\varepsilon)_+$. Since X^ε is a compound Poisson process,

$$g_t(X_t^\varepsilon) - g_t(0) = \sum_{0<s\leq t} \left(g_t(X_s^\varepsilon) - g_t(X_{s-}^\varepsilon)\right),$$

so that, due to the compensation formula (cf. Proposition 2.2),

$$\begin{aligned}\mathbb{E}\left(X_t^\varepsilon - tC_\varepsilon\right)_+ &= g_t(0) + \mathbb{E}\left(\sum_{0\leq s\leq t} \left(g_t(X_s^\varepsilon) - g_t(X_{s-}^\varepsilon)\right)\right) \\ &= t\left(-C_\varepsilon\right)_+ + \mathbb{E}\left(\int_0^t ds \int_{\{\varepsilon\leq|y|\leq 1\}} \left(g_t(X_s^\varepsilon + y) - g_t(X_s^\varepsilon)\right)\nu(dy)\right).\end{aligned}$$

For any fixed $\varepsilon \in (0,1)$, we have

$$\begin{aligned}\mathbb{E}\left(\int_0^t ds \int_{\{\varepsilon\leq|y|\leq 1\}} \left(g_t(X_s^\varepsilon + y) - g_t(X_s^\varepsilon)\right)\nu(dy)\right) &= \mathbb{E}\left[\int_0^t ds \int \left[(X_s^\varepsilon + y)_+ - (X_s^\varepsilon)_+\right]\nu(dy)\right] + o(t) \\ &= t\int_{\{\varepsilon\leq|y|\leq 1\}} y_+\nu(dy) + o(t).\end{aligned}$$

Therefore,

$$\mathbb{E}[(X_t^\varepsilon - tC_\varepsilon)_+] = t\left((-C_\varepsilon)_+ + \int_{\{\varepsilon\leq|y|\leq 1\}} y_+\nu(dy)\right) + o(t).$$

Going back to (5.2), we derive

$$\begin{aligned}\liminf_{t\to 0}\mathbb{E}\left[\left(\frac{\tilde{X}_t^0}{t}\right)_+\right] &\geq (-C_\varepsilon)_+ + \int_{\{\varepsilon\leq|y|\leq 1\}} y_+\nu(dy)\\ &= \left(-\int_{\{\varepsilon\leq|y|\leq 1\}} y\nu(dy)\right)_+ + \int_{\{\varepsilon\leq|y|\leq 1\}} y_+\nu(dy).\end{aligned}$$

Since $\int_{\{|y|\leq 1\}}|y|\nu(dy)=\infty$, we have either $\lim_{\varepsilon\downarrow 0}\int_{\{\varepsilon\leq|y|\leq 1\}} y_+\nu(dy)=\infty$ or

$$\lim_{\varepsilon\downarrow 0}\left(-\int_{\{\varepsilon\leq|y|\leq 1\}} y\nu(dy)\right)_+ = \infty,$$

and (5.1) follows. ◇

6 Critical price and tempered stable processes

Throughout this section, the following assumption is in force.

(AS) We have

$$\mathbb{E}\left(e^{iuX_t}\right) = \exp\left(t\int\left(e^{iuy}-1-iu(e^y-1)\right)\nu(dy)\right),$$

with $\int(e^y-1)_+\nu(dy) < r-\delta$ and, for some $a_0<0$,

$$\mathbb{1}_{\{a_0<y<0\}}\nu(dy) = \frac{\eta(y)}{|y|^{1+\alpha}}\mathbb{1}_{\{a_0<y<0\}}dy,$$

where $1<\alpha<2$ and η is a positive bounded Borel measurable function on $[a_0,0)$, which satisfies $\lim_{y\to 0}\eta(y)=\eta_0>0$.

Note that, under this assumption, we have $\nu[(-\infty,0)]>0$, so that (2.13) is satisfied.

Theorem 6.1 *Under assumption (AS), we have*

$$\lim_{t\to T}\frac{K-b(t)}{(T-t)^{1/\alpha}|\ln(T-t)|^{1-\frac{1}{\alpha}}} = K\left(\eta_0\frac{\Gamma(2-\alpha)}{\alpha-1}\right)^{1/\alpha}.$$

Remark 6.2 Our assumptions exclude the case $\alpha=1$. In this case, we still have a Lévy process with infinite variation and we can apply Theorem 5.1: $\lim_{t\to T}(K-b(t))/(T-t)=\infty$. On the other hand, using comparison arguments, we can deduce from Theorem 6.1 that $\lim_{t\to T}(K-b(t))/(T-t)^{1-\varepsilon}=0$, for all $\varepsilon>0$. It would be interesting to clarify the rate of convergence of $b(t)$ to K in this case. As pointed out by the referees, the case $\alpha=1$ is important in reference to the Normal Inverse Gaussian model. However, in this model, the Lévy measure is symmetric, and the assumption $r-\delta-\int(e^y-1)_+\nu(dy)\geq 0$ is not satisfied, so that $\lim_{t\to T}b(t)<K$ (see Theorem 2.6). The asymptotics of $b(t)$ near T cannot then be treated by the methods of the present paper (see [12] for the Black-Scholes case).

For the proof of Theorem 6.1, we use the same approach as in Section 4. Namely, we first characterise the rate of convergence of the *European* critical price $b_e(t)$: this is done in Section 6.1 (see Proposition 6.4, and recall that $b(t) \leq b_e(t) \leq K$). Then, we estimate the difference between the European and the American critical prices. In fact, Theorem 6.1 is a direct consequence of Proposition 6.4, combined with Proposition 6.9.

Before investigating the behavior of the European critical price, we establish a crucial consequence of assumption (AS), namely the fact that, for small t, the Lévy process at time t behaves asymptotically like a one-sided stable random variable of order α.

Lemma 6.3 *Under assumption (AS), as t goes to 0, the random variable $X_t/t^{1/\alpha}$ converges in distribution to a random variable Z with characteristic function given by*

$$\mathbb{E}\left(e^{iuZ}\right) = \exp\left(\eta_0 \int_0^{+\infty} \left(e^{-iuz} - 1 + iuz\right) \frac{dz}{z^{1+\alpha}}\right), \quad u \in \mathbb{R}.$$

Proof: Introduce the following decomposition of the process X

$$X_t = X_t^0 - t\int (e^y - 1)_+ \nu(dy) + \bar{X}_t, \quad t \geq 0,$$

where

$$X_t^0 = \sum_{0<s\leq t} \Delta X_s \mathbb{1}_{\{\Delta X_s > 0\}}.$$

Note that the process X^0 is well defined because $\int y_+ \nu(dy) \leq \int (e^y - 1)_+ \nu(dy) < \infty$, and the characteristic function of $\bar{X}_t$ is given by

$$\mathbb{E}\left(e^{iu\bar{X}_t}\right) = \exp\left(t\int_{(-\infty,0)} \left(e^{iuy} - 1 - iu(e^y - 1)\right)\nu(dy)\right), \quad u \in \mathbb{R}. \tag{6.1}$$

We have $\lim_{t\downarrow 0} \frac{X_t^0}{t} = 0$ a.s. (see [17], Section 47), so that, with probability one,

$$\lim_{t\downarrow 0} \frac{X_t - \bar{X}_t}{t^{1/\alpha}} = 0.$$

We will now prove that $\bar{X}_t/t^{1/\alpha}$ weakly converges to Z as $t \to 0$. For a fixed $u \in \mathbb{R}$, we have

$$\mathbb{E}\left(e^{iu\frac{\bar{X}_t}{t^{1/\alpha}}}\right) = \exp\left(t\int_{(-\infty,0)} \left(e^{iuy/t^{1/\alpha}} - 1 - \frac{iu}{t^{1/\alpha}}(e^y - 1)\right)\nu(dy)\right).$$

The integral in the exponential can be split in two parts

$$\begin{aligned}\int_{(-\infty,0)} \left(e^{iuy/t^{1/\alpha}} - 1 - \frac{iu}{t^{1/\alpha}}(e^y - 1)\right)\nu(dy) = &\int_{(-\infty,a_0]} \left(e^{iuy/t^{1/\alpha}} - 1 - \frac{iu}{t^{1/\alpha}}(e^y - 1)\right)\nu(dy) \\ &+ \int_{a_0}^{0} \left(e^{iuy/t^{1/\alpha}} - 1 - \frac{iu}{t^{1/\alpha}}(e^y - 1)\right)\nu(dy).\end{aligned}$$

We have

$$\left|\int_{(-\infty,a_0]}\left(e^{iuy/t^{1/\alpha}}-1-\frac{iu}{t^{1/\alpha}}(e^y-1)\right)\nu(dy)\right|\leq 2\nu((-\infty,a_0])+\frac{|u|}{t^{1/\alpha}}\int_{(-\infty,a_0]}|e^y-1|\nu(dy),$$

so that

$$\lim_{t\downarrow 0}\left(t\int_{(-\infty,a_0]}\left(e^{iuy/t^{1/\alpha}}-1-\frac{iu}{t^{1/\alpha}}(e^y-1)\right)\nu(dy)\right)=0.$$

On the other hand,

$$\begin{aligned}\int_{a_0}^{0}\left(e^{iuy/t^{1/\alpha}}-1-\frac{iu}{t^{1/\alpha}}(e^y-1)\right)\nu(dy) &= \int_{a_0}^{0}\left(e^{iuy/t^{1/\alpha}}-1-\frac{iu}{t^{1/\alpha}}y\right)\frac{\eta(y)}{|y|^{1+\alpha}}dy\\ &\quad+\frac{iu}{t^{1/\alpha}}\int_{a_0}^{0}\left(y-(e^y-1)\right)\frac{\eta(y)}{|y|^{1+\alpha}}dy\\ &= \int_{0}^{|a_0|}\left(e^{-iuy/t^{1/\alpha}}-1+\frac{iu}{t^{1/\alpha}}y\right)\frac{\eta(-y)}{y^{1+\alpha}}dy\\ &\quad+O(t^{-1/\alpha}),\end{aligned}$$

where the last equality follows from the boundedness of η and the fact that $\int_{(a_0,0)} y^2\nu(dy)<\infty$. Hence, using the substitution $z=y/t^{1/\alpha}$,

$$\begin{aligned}\int_{a_0}^{0}\left(e^{iuy/t^{1/\alpha}}-1-\frac{iu}{t^{1/\alpha}}(e^y-1)\right)\nu(dy) &= \frac{1}{t}\int_{0}^{\frac{|a_0|}{t^{1/\alpha}}}\left(e^{-iuz}-1+iuz\right)\frac{\eta(-z/t^{1/\alpha})}{z^{1+\alpha}}dz\\ &\quad+O(t^{-1/\alpha}),\end{aligned}$$

so that, by dominated convergence,

$$\lim_{t\downarrow 0}\left(t\int_{a_0}^{0}\left(e^{iuy/t^{1/\alpha}}-1-\frac{iu}{t^{1/\alpha}}(e^y-1)\right)\nu(dy)\right)=\int_{0}^{+\infty}\left(e^{-iuz}-1+iuz\right)\frac{\eta_0}{z^{1+\alpha}}dz,$$

and the lemma is proved. ◇

6.1 European critical price

Denote $\theta=T-t$. The equality $P_e(t,b_e(t))=K-b_e(t)$ can be written as follows

$$\begin{aligned}K-b_e(t) &= \mathbb{E}e^{-r\theta}\left(K-b_e(t)e^{(r-\delta)\theta+X_\theta}\right)_+\\ &= Ke^{-r\theta}-b_e(t)e^{-\delta\theta}+\mathbb{E}e^{-r\theta}\left(b_e(t)e^{(r-\delta)\theta+X_\theta}-K\right)_+.\end{aligned}$$

Hence

$$\frac{K}{b_e(t)}\left(1-e^{-r\theta}\right)-\left(1-e^{-\delta\theta}\right)=\mathbb{E}e^{-r\theta}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+. \tag{6.2}$$

Since $\lim_{t\to T} b_e(t) = K$, the left-hand side is equal to $(r-\delta)\theta + o(\theta)$. For the study of the right-hand side, let $\zeta(\theta) = \frac{K}{b_e(t)} - 1$, so that from (6.2) we derive

$$\begin{aligned}(r-\delta)\theta &= \mathbb{E}e^{-r\theta}\left(e^{(r-\delta)\theta + X_\theta} - 1 - \zeta(\theta)\right)_+ + o(\theta)\\ &= \mathbb{E}\left(e^{(r-\delta)\theta + X_\theta} - 1 - \zeta(\theta)\right)_+ + o(\theta), \qquad (6.3)\end{aligned}$$

where we have used the fact that $\lim_{\theta\to 0}\mathbb{E}\left(e^{(r-\delta)\theta + X_\theta} - 1 - \zeta(\theta)\right)_+ = 0$. The following statement clarifies the behavior of $\zeta(\theta)$ as $\theta \downarrow 0$.

Proposition 6.4 *Under assumption (AS), we have*

$$\lim_{\theta\downarrow 0}\frac{\zeta(\theta)}{\theta^{1/\alpha}|\ln\theta|^{1-\frac{1}{\alpha}}} = (\alpha\eta_0 I_\alpha)^{1/\alpha} = \left(\eta_0\frac{\Gamma(2-\alpha)}{\alpha-1}\right)^{1/\alpha},$$

where

$$I_\alpha = \int_0^{+\infty}\left(e^{-z} - 1 + z\right)\frac{dz}{z^{1+\alpha}} = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}.$$

The first step in the proof of Proposition 6.4 is the following lemma.

Lemma 6.5 *We have*

$$\lim_{\theta\to 0}\frac{\zeta(\theta)}{\theta^{1/\alpha}} = +\infty. \qquad (6.4)$$

Proof: Note that

$$\left|\mathbb{E}\left(e^{(r-\delta)\theta + X_\theta} - 1 - \zeta(\theta)\right)_+ - \mathbb{E}\left(e^{X_\theta} - 1 - \zeta(\theta)\right)_+\right| \le \left(e^{(r-\delta)\theta} - 1\right)\mathbb{E}\left(e^{X_\theta}\right) = O(\theta),$$

so that, in view of (6.3), we have

$$\mathbb{E}\left(e^{X_\theta} - 1 - \zeta(\theta)\right)_+ = O(\theta).$$

Since $e^x \ge 1 + x$, we also have $\mathbb{E}\left(X_\theta - \zeta(\theta)\right)_+ = O(\theta)$. Therefore

$$\lim_{\theta\downarrow 0}\mathbb{E}\left(\frac{X_\theta}{\theta^{1/\alpha}} - \frac{\zeta(\theta)}{\theta^{1/\alpha}}\right)_+ = 0.$$

If we had $\liminf_{\theta\downarrow 0}\zeta(\theta)/\theta^{1/\alpha} = \lambda \in [0, +\infty)$, we would deduce from Lemma 6.3 and Fatou's Lemma that

$$\mathbb{E}\left(Z - \lambda\right)_+ = 0.$$

Hence $\mathbb{P}(Z \le \lambda) = 1$. However, the support of the random variable Z (which is a one-sided stable random variable of order α) is the whole real line. This proves (6.4) by contradiction. $\diamond$

The next lemma provides some estimates for the moment generating function of the process $\bar{X}$, defined by

$$\bar{X}_t = X_t + t\int (e^y - 1)_+\nu(dy) - \sum_{0<s\le t}\Delta X_s \mathbb{1}_{\{\Delta X_s > 0\}}, \quad t \ge 0.$$

Lemma 6.6 *We have, for all $\rho \geq 0$, $t \geq 0$,*

$$\mathbb{E}\left(e^{\rho \bar{X}_t}\right) = e^{t\bar{\varphi}(\rho)},$$

with

$$\bar{\varphi}(\rho) = \int_{(-\infty,0)} \left(e^{\rho y} - 1 - \rho(e^y - 1)\right) \nu(dy), \quad \rho \geq 0. \tag{6.5}$$

Morover, for any $a \in [a_0, 0)$ and any $\rho \geq 0$, we have

$$\rho^\alpha H_a(\rho) - \nu_a - \rho\bar{\nu}_a \leq \bar{\varphi}(\rho) \leq \rho\nu_a + \rho^\alpha H_a(\rho),$$

where

$$\nu_a = \nu((-\infty, a]), \quad \bar{\nu}_a = \int_0^{|a|} \left(e^{-y} - 1 + y\right) \nu(dy)$$

and

$$H_a(\rho) = \int_0^{|a|\rho} \left(e^{-z} - 1 + z\right)\right) \frac{\eta(-z/\rho)}{z^{1+\alpha}} dz.$$

Proof: First, note that (6.5) is deduced from (6.1) by analytic continuation. Now, fix $\rho \geq 0$ and $a \in [a_0, 0)$. We have,

$$\bar{\varphi}(\rho) = \int_{(-\infty,a]} \left(e^{\rho y} - 1 - \rho(e^y - 1)\right) \nu(dy) + \bar{\varphi}_a(\rho)$$

with the notation

$$\bar{\varphi}_a(\rho) = \int_a^0 \left(e^{\rho y} - 1 - \rho(e^y - 1)\right) \nu(dy), \quad \rho \geq 0.$$

For $y \in (-\infty, a]$, we have $-1 \leq e^{\rho y} - 1 - \rho(e^y - 1) \leq \rho$. Therefore

$$\bar{\varphi}_a(\rho) - \nu_a \leq \bar{\varphi}(\rho) \leq \bar{\varphi}_a(\rho) + \rho\nu_a.$$

On the other hand,

$$\begin{aligned}
\bar{\varphi}_a(\rho) &= \int_0^{|a|} \left(e^{-\rho y} - 1 - \rho(e^{-y} - 1)\right) \frac{\eta(-y)}{y^{1+\alpha}} dy \\
&= \int_0^{|a|} \left(e^{-\rho y} - 1 + \rho y)\right) \frac{\eta(-y)}{y^{1+\alpha}} dy - \rho \int_0^{|a|} \left(e^{-y} - 1 + y\right) \frac{\eta(-y)}{y^{1+\alpha}} dy.
\end{aligned}$$

We have $e^{-y} - 1 + y \geq 0$. Hence

$$-\rho \int_0^{|a|} \left(e^{-y} - 1 + y\right) \frac{\eta(-y)}{y^{1+\alpha}} dy + \psi_a(\rho) \leq \bar{\varphi}_a(\rho) \leq \psi_a(\rho),$$

where

$$\begin{aligned}
\psi_a(\rho) &= \int_0^{|a|} \left(e^{-\rho y} - 1 + \rho y)\right) \frac{\eta(-y)}{y^{1+\alpha}} dy \\
&= \rho^\alpha \int_0^{|a|\rho} \left(e^{-z} - 1 + z)\right) \frac{\eta(-z/\rho)}{z^{1+\alpha}} dz = \rho^\alpha H_a(\rho).
\end{aligned}$$

◇

The crucial step in the proof of Proposition 6.4 is an asymptotic estimate for the tail of the distribution of $\bar{X}_\theta/\theta^{1/\alpha}$ as θ approaches 0. This will be given in Lemma 6.8. We first give a preliminary uniform bound.

Lemma 6.7 *Let $a \in [a_0, 0)$. There exists a positive constant C_a such that, for all $\theta > 0$, $t > 0$, we have*

$$\ln \mathbb{P}\left(\frac{\bar{X}_\theta}{\theta^{1/\alpha}} \geq t\right) \leq C_a \theta^{1-\frac{1}{\alpha}} t^{\frac{1}{\alpha-1}} - J_\alpha(a) t^{\frac{\alpha}{\alpha-1}},$$

where

$$J_\alpha(a) = \frac{\alpha - 1}{\alpha^{\frac{\alpha}{\alpha-1}} \left(\eta^*(a) I_\alpha\right)^{\frac{1}{\alpha-1}}},$$

with

$$\eta^*(a) = \sup_{u \in (a,0)} \eta(u) \quad \textit{and} \quad I_\alpha = \int_0^{+\infty} \left(e^{-z} - 1 + z\right) \frac{dz}{z^{1+\alpha}} = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}.$$

Proof: For any $p > 0$, we have, using Markov's inequality and Lemma 6.6,

$$\begin{aligned}
\mathbb{P}\left(\frac{\bar{X}_\theta}{\theta^{1/\alpha}} \geq t\right) &\leq e^{-pt} \mathbb{E}\left(e^{p\bar{X}_\theta/\theta^{1/\alpha}}\right) \\
&= e^{-pt} e^{\theta \bar{\varphi}\left(p/\theta^{1/\alpha}\right)} \\
&\leq e^{-pt} e^{p^\alpha H_a(p/\theta^{1/\alpha}) + \theta^{1-\frac{1}{\alpha}} p \nu_a} \\
&\leq e^{-pt} e^{p^\alpha \eta^*(a) I_\alpha + \theta^{1-\frac{1}{\alpha}} p \nu_a},
\end{aligned}$$

where the last inequality follows from $H_a(\rho) \leq \eta^*(a) I_\alpha$. By choosing $p = \left(\frac{t}{\alpha \eta^*(a) I_\alpha}\right)^{1/(\alpha-1)}$, we get

$$\mathbb{P}\left(\frac{\bar{X}_\theta}{\theta^{1/\alpha}} \geq t\right) \leq \exp\left(-J_\alpha(a) t^{\frac{\alpha}{\alpha-1}} + C_a \theta^{1-\frac{1}{\alpha}} t^{\frac{1}{\alpha-1}}\right),$$

with $C_a = \left(\alpha \eta^*(a) I_\alpha\right)^{-\frac{1}{\alpha-1}} \nu_a$. ◇

We are now in a position to prove the main estimate for the proof of Proposition 6.4.

Lemma 6.8 *Denote, for $\theta > 0$, $\bar{Z}_\theta = \dfrac{\bar{X}_\theta}{\theta^{1/\alpha}}$. We have, for any function $\xi : (0, +\infty) \to (0, +\infty)$ satisfying $\lim_{\theta \downarrow 0} \xi(\theta) = +\infty$,*

$$\lim_{\theta \downarrow 0} \frac{\ln \mathbb{P}(\bar{Z}_\theta \geq \xi(\theta))}{(\xi(\theta))^{\frac{\alpha}{\alpha-1}}} = -J_\alpha(0), \ \textit{where } J_\alpha(0) = \lim_{a \uparrow 0} J_\alpha(a) = \frac{\alpha - 1}{\left(\alpha^\alpha \eta_0 I_\alpha\right)^{\frac{1}{\alpha-1}}}.$$

Proof: We first prove

$$\limsup_{\theta\downarrow 0}\frac{\ln\mathbb{P}(\bar{Z}_\theta\geq\xi(\theta))}{(\xi(\theta))^{\frac{\alpha}{\alpha-1}}}\leq -J_\alpha(0). \tag{6.6}$$

Applying Lemma 6.7 with $t=\xi(\theta)$, we have, for all $a\in[a_0,0)$,

$$\ln\mathbb{P}(\bar{Z}_\theta\geq\xi(\theta))\leq -J_\alpha(a)(\xi(\theta))^{\frac{\alpha}{\alpha-1}}+C_a\theta^{1-\frac{1}{\alpha}}(\xi(\theta))^{\frac{1}{\alpha-1}}.$$

Hence

$$\limsup_{\theta\downarrow 0}\frac{\ln\mathbb{P}(\bar{Z}_\theta\geq\xi(\theta))}{(\xi(\theta))^{\frac{\alpha}{\alpha-1}}}\leq -J_\alpha(a),$$

and (6.6) follows by letting a go to 0.

In order to derive a lower bound for the lim inf, we proceed as follows. Given any $p>0$ and any $t>0$, we have

$$\begin{aligned}
\mathbb{E}\left(e^{p\bar{Z}_\theta}\right)&=\mathbb{E}\left(e^{p\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta<t\}}\right)+\mathbb{E}\left(e^{p\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq t\}}\right)\\
&=\mathbb{E}\left(\int_{-\infty}^{\bar{Z}_\theta}pe^{ps}ds\mathbb{1}_{\{\bar{Z}_\theta<t\}}\right)+\mathbb{E}\left(e^{p\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq t\}}\right)\\
&\leq 1+\mathbb{E}\left(\int_0^{+\infty}pe^{ps}\mathbb{1}_{\{0<s\leq\bar{Z}_\theta<t\}}ds\right)+\mathbb{E}\left(e^{p\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq t\}}\right)\\
&\leq 1+\int_0^t pe^{ps}\mathbb{P}\left(\bar{Z}_\theta\geq s\right)ds+\mathbb{E}\left(e^{p\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq t\}}\right).
\end{aligned}$$

It follows from Lemma 6.7 that $\mathbb{P}\left(\bar{Z}_\theta\geq s\right)\leq\exp\left(C_a\theta^{1-\frac{1}{\alpha}}s^{\frac{1}{\alpha-1}}-J_\alpha(a)s^{\frac{\alpha}{\alpha-1}}\right)$, so that

$$\mathbb{E}\left(e^{p\bar{Z}_\theta}\right)\leq 1+pF_a(\theta,t)\int_0^t e^{ps-J_\alpha(a)s^{\frac{\alpha}{\alpha-1}}}ds+\mathbb{E}\left(e^{p\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq t\}}\right), \tag{6.7}$$

with

$$F_a(\theta,t)=e^{C_a\theta^{1-\frac{1}{\alpha}}t^{\frac{1}{\alpha-1}}}.$$

For notational convenience, let

$$\hat{\alpha}=\frac{\alpha}{\alpha-1},\quad\text{so that}\quad\hat{\alpha}-1=\frac{1}{\alpha-1},$$

and

$$f_p(s)=ps-J_\alpha(a)s^{\hat{\alpha}},\quad s>0.$$

We have $f_p'(s)=p-\hat{\alpha}J_\alpha(a)s^{\hat{\alpha}-1}$, so that the function f_p is increasing on the interval $[0,s_p^*]$ and decreasing on $[s_p^*,+\infty)$, where

$$s_p^*=\left(\frac{p}{J_\alpha(a)\hat{\alpha}}\right)^{\frac{1}{\hat{\alpha}-1}}=\left(\frac{p}{J_\alpha(a)\hat{\alpha}}\right)^{\alpha-1}.$$

We now fix $t > 0$ and choose $p = Mt^{\frac{1}{\alpha-1}}$, where M is a constant satisfying

$$M \;>\; J_\alpha(0)\hat{\alpha} \;=\; \frac{\alpha}{\alpha-1}J_\alpha(0) \;=\; \frac{1}{(\alpha\eta_0 I_\alpha)^{\frac{1}{\alpha-1}}}.$$

We then have $M > J_\alpha(a)\hat{\alpha}$ for all $a \in [a_0, 0)$, so that

$$t < \left(\frac{M}{J_\alpha(a)\hat{\alpha}}\right)^{\alpha-1} t \;=\; \left(\frac{p}{J_\alpha(a)\hat{\alpha}}\right)^{\alpha-1} = s_p^*.$$

Therefore

$$\forall s \in [0,t], \quad f_p(s) \le f_p(t) = pt - J_\alpha(a)t^{\hat{\alpha}} = t^{\hat{\alpha}}\left(M - J_\alpha(a)\right),$$

so that

$$\int_0^t e^{f_p(s)}ds \le te^{f_p(t)} = t\exp\left(t^{\hat{\alpha}}\left(M - J_\alpha(a)\right)\right).$$

Going back to (6.7), we get

$$\begin{aligned}\mathbb{E}\left(e^{Mt^{\frac{1}{\alpha-1}}\bar{Z}_\theta}\right) &\le 1 + Mt^{\frac{1}{\alpha-1}}F_a(\theta,t)t\exp\left(t^{\hat{\alpha}}\left(M - J_\alpha(a)\right)\right) + \mathbb{E}\left(e^{Mt^{\frac{1}{\alpha-1}}\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta \ge t\}}\right)\\ &= 1 + Mt^{\hat{\alpha}}F_a(\theta,t)\exp\left(t^{\hat{\alpha}}\left(M - J_\alpha(a)\right)\right) + \mathbb{E}\left(e^{Mt^{\hat{\alpha}-1}\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta \ge t\}}\right). \qquad (6.8)\end{aligned}$$

On the other hand, we have, using Lemma 6.6,

$$\begin{aligned}\mathbb{E}\left(e^{Mt^{\frac{1}{\alpha-1}}\bar{Z}_\theta}\right) &= \mathbb{E}\left(e^{Mt^{\hat{\alpha}-1}\bar{X}_\theta/\theta^{1/\alpha}}\right)\\ &= \exp\left(\theta\bar{\varphi}(Mt^{\hat{\alpha}-1}/\theta^{1/\alpha})\right)\\ &\ge \exp\left[\theta\left(\left(\frac{Mt^{\hat{\alpha}-1}}{\theta^{1/\alpha}}\right)^\alpha H_a(Mt^{\hat{\alpha}-1}/\theta^{1/\alpha}) - \nu_a - \frac{Mt^{\hat{\alpha}-1}}{\theta^{1/\alpha}}\bar{\nu}_a\right)\right]\\ &= \exp\left(t^{\hat{\alpha}}K_a(M,\theta,t)\right)G_a(M,\theta,t), \qquad (6.9)\end{aligned}$$

where

$$K_a(M,\theta,t) = M^\alpha H_a(Mt^{\hat{\alpha}-1}/\theta^{1/\alpha}) \quad \text{and} \quad G_a(M,\theta,t) = e^{-\theta\nu_a - M\theta^{1-\frac{1}{\alpha}}t^{\hat{\alpha}-1}\bar{\nu}_a}.$$

Combining (6.8) and (6.9), we have

$$\begin{aligned}\mathbb{E}\left(e^{Mt^{\hat{\alpha}-1}\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta \ge t\}}\right) \ge e^{t^{\hat{\alpha}}K_a(M,\theta,t)}&\Big(G_a(M,\theta,t) - e^{-t^{\hat{\alpha}}K_a(M,\theta,t)}\\ &-Mt^{\hat{\alpha}}F_a(\theta,t)\exp\left(t^{\hat{\alpha}}\left(M - J_\alpha(a) - K_a(M,\theta,t)\right)\right)\Big). \qquad (6.10)\end{aligned}$$

In order to study the sign of the quantity $M - J_\alpha(a) - K_a(M,\theta,t)$, which is equal to $M - M^\alpha H_a(Mt^{\hat{\alpha}-1}/\theta^{1/\alpha}) - J_\alpha(a)$, we introduce the function

$$\psi_\alpha(M) = M - M^\alpha\eta_0 I_\alpha, \quad M > 0.$$

We have

$$\psi'_\alpha(M) = 1 - \alpha M^{\alpha-1}\eta_0 I_\alpha.$$

Since $M > \frac{1}{(\alpha\eta_0 I_\alpha)^{\frac{1}{\alpha-1}}}$, we have $\psi'_\alpha(M) < 0$. Therefore

$$\begin{aligned}\psi_\alpha(M) &< \psi_\alpha\left(J_\alpha(0)\hat{\alpha}\right)\\ &= \psi_\alpha\left(\frac{1}{(\alpha\eta_0 I_\alpha)^{\frac{1}{\alpha-1}}}\right)\\ &= \frac{1}{(\alpha\eta_0 I_\alpha)^{\frac{1}{\alpha-1}}}\left(1-\frac{1}{\alpha}\right) = J_\alpha(0).\end{aligned}$$

Since $\lim_{a\uparrow 0} J_\alpha(a) = J_\alpha(0)$, we also have, for a close to 0,

$$\psi_\alpha(M) < J_\alpha(a).$$

Now, consider any function $\xi : (0,+\infty) \to (0,+\infty)$, such that $\lim_{\theta\downarrow 0}\xi(\theta) = +\infty$. We will apply (6.10) with $t = \xi(\theta)$. Note that $\lim_{\theta\downarrow 0} K_a(M,\theta,\xi(\theta)) = M^\alpha\eta_0 I_\alpha$, so that

$$\lim_{\theta\downarrow 0}\left(M - J_\alpha(a) - K_a(M,\theta,\xi(\theta))\right) = \psi_\alpha(M) - J_\alpha(a) < 0,$$

and

$$\lim_{\theta\downarrow 0}\left(\xi^{\hat{\alpha}}(\theta)\frac{F_a(\theta,\xi(\theta))}{G_a(M,\theta,\xi(\theta))}\exp\left[\xi^{\hat{\alpha}}(\theta)\left(M - J_\alpha(a) - K_a(M,\theta,\xi(\theta))\right)\right]\right) = 0.$$

For the last equality, we observe that $\frac{F_a(\theta,\xi(\theta))}{G_a(M,\theta,\xi(\theta))}$ behaves like $e^{C\theta^{(\alpha-1)/\alpha}\xi^{\hat{\alpha}-1}(\theta)}$, for some $C > 0$. Therefore, we deduce from (6.10) that

$$\liminf_{\theta\downarrow 0}\frac{\ln\mathbb{E}\left(e^{M\xi^{\hat{\alpha}-1}(\theta)\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq\xi(\theta)\}}\right)}{\xi^{\hat{\alpha}}(\theta)} \geq \lim_{\theta\downarrow 0} K_a(M,\theta,\xi(\theta)) = M^\alpha\eta_0 I_\alpha. \tag{6.11}$$

Now, it follows from Hölder's inequality that, for any $q > 1$,

$$\begin{aligned}\mathbb{E}\left(e^{M\xi^{\hat{\alpha}-1}(\theta)\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq\xi(\theta)\}}\right) &\leq \left(\mathbb{E}\left(e^{qM\xi^{\hat{\alpha}-1}(\theta)\bar{Z}_\theta}\right)\right)^{1/q}\left[\mathbb{P}\left(\bar{Z}_\theta\geq\xi(\theta)\right)\right]^{1-\frac{1}{q}}\\ &= \exp\left[\frac{\theta}{q}\bar{\varphi}\left(qM\xi^{\hat{\alpha}-1}(\theta)/\theta^{1/\alpha}\right)\right]\left[\mathbb{P}\left(\bar{Z}_\theta\geq\xi(\theta)\right)\right]^{1-\frac{1}{q}}.\end{aligned}$$

Hence

$$\begin{aligned}\left(1-\frac{1}{q}\right)\ln\mathbb{P}\left(\bar{Z}_\theta\geq\xi(\theta)\right) &\geq \ln\left[\mathbb{E}\left(e^{M\xi^{\hat{\alpha}-1}(\theta)\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq\xi(\theta)\}}\right)\right] - \frac{\theta}{q}\bar{\varphi}\left(q\frac{M\xi^{\hat{\alpha}-1}(\theta)}{\theta^{1/\alpha}}\right)\\ &\geq \ln\left[\mathbb{E}\left(e^{M\xi^{\hat{\alpha}-1}(\theta)\bar{Z}_\theta}\mathbb{1}_{\{\bar{Z}_\theta\geq\xi(\theta)\}}\right)\right]\\ &\quad -\nu_a\theta^{1-\frac{1}{\alpha}}M\xi^{\hat{\alpha}-1}(\theta) - M^\alpha\xi^{\hat{\alpha}}(\theta)q^{\alpha-1}H_a\left(\frac{qM\xi^{\hat{\alpha}-1}(\theta)}{\theta^{1/\alpha}}\right),\end{aligned}$$

where the last inequality follows from Lemma 6.6. We now deduce from (6.11) and from the fact that $\lim_{\rho\to\infty} H_a(\rho) = \eta_0 I_\alpha$

$$\left(1-\frac{1}{q}\right)\liminf_{\theta\downarrow 0}\frac{\ln\mathbb{P}\left(\bar{Z}_\theta \geq \xi(\theta)\right)}{\xi^{\hat{\alpha}}(\theta)} \geq M^\alpha \eta_0 I_\alpha \left(1-q^{\alpha-1}\right).$$

Hence

$$\liminf_{\theta\downarrow 0}\frac{\ln\mathbb{P}\left(\bar{Z}_\theta \geq \xi(\theta)\right)}{\xi^{\hat{\alpha}}(\theta)} \geq M^\alpha \eta_0 I_\alpha \frac{q-q^\alpha}{q-1},$$

and, by taking the limit as q goes to 1,

$$\liminf_{\theta\downarrow 0}\frac{\ln\mathbb{P}\left(\bar{Z}_\theta \geq \xi(\theta)\right)}{\xi^{\hat{\alpha}}(\theta)} \geq -(\alpha-1)M^\alpha \eta_0 I_\alpha$$

Since M is arbitrary in $\left((\alpha\eta_0 I_\alpha)^{\frac{-1}{\alpha-1}}, +\infty\right)$, we can take the limit as M goes to $(\alpha\eta_0 I_\alpha)^{\frac{-1}{\alpha-1}}$, so that

$$\liminf_{\theta\downarrow 0}\frac{\ln\mathbb{P}\left(\bar{Z}_\theta \geq \xi(\theta)\right)}{\xi^{\hat{\alpha}}(\theta)} \geq -(\alpha-1)\frac{\eta_0 I_\alpha}{(\alpha\eta_0 I_\alpha)^{\frac{\alpha}{\alpha-1}}} = -J_\alpha(0).$$

◇

Proof of Proposition 6.4: We first prove

$$\liminf_{\theta\downarrow 0}\frac{\zeta(\theta)}{\theta^{1/\alpha}|\ln\theta|^{1-\frac{1}{\alpha}}} \geq \left(\alpha\eta_0 I_0\right)^{1/\alpha}. \tag{6.12}$$

We deduce from (6.3) that

$$\mathbb{E}\left(e^{(r-\delta)\theta+X_\theta}-1-\zeta(\theta)\right)_+ = (r-\delta)\theta + o(\theta).$$

We have the decomposition $X_\theta = \bar{X}_\theta + X^0_\theta - \theta\int(e^y-1)_+\nu(dy)$, where the processes $\bar{X}$ and X^0 are independent and $\mathbb{E}\left(e^{X^0_\theta}\right) = e^{\theta\int(e^y-1)_+\nu(dy)}$, so that, by conditioning with respect to $\bar{X}$,

$$\begin{aligned}\mathbb{E}\left(e^{(r-\delta)\theta+X_\theta}-1-\zeta(\theta)\right)_+ &\geq \mathbb{E}\left(e^{(r-\delta)\theta+\bar{X}_\theta}-1-\zeta(\theta)\right)_+\\ &\geq \mathbb{E}\left((r-\delta)\theta+\bar{X}_\theta-\zeta(\theta)\right)_+ \geq \mathbb{E}\left(\bar{X}_\theta-\zeta(\theta)\right)_+.\end{aligned}$$

Hence, with the notations of Lemma 6.8,

$$\mathbb{E}\left(\bar{Z}_\theta-\bar{\zeta}(\theta)\right)_+ \leq (r-\delta)\theta^{1-\frac{1}{\alpha}} + o(\theta^{1-\frac{1}{\alpha}}),\ \text{where } \bar{\zeta}(\theta) = \frac{\zeta(\theta)}{\theta^{1/\alpha}}.$$

We deduce thereof that there exists a positive constant C such that, for θ close to 0,

$$\ln\mathbb{E}\left(\bar{Z}_\theta-\bar{\zeta}(\theta)\right)_+ \leq \left(1-\frac{1}{\alpha}\right)\ln\theta + C. \tag{6.13}$$

Now, given any $\beta > 1$, we have

$$\mathbb{E}\left(\bar{Z}_\theta - \bar{\zeta}(\theta)\right)_+ \geq (\beta-1)\bar{\zeta}(\theta)\mathbb{P}\left(\bar{Z}_\theta \geq \beta\bar{\zeta}(\theta)\right),$$

so that

$$\ln \mathbb{E}\left(\bar{Z}_\theta - \bar{\zeta}(\theta)\right)_+ \geq \ln[(\beta-1)\bar{\zeta}(\theta)] + \ln \mathbb{P}\left(\bar{Z}_\theta \geq \beta\bar{\zeta}(\theta)\right).$$

Since $\lim_{\theta\downarrow 0}\bar{\zeta}(\theta) = +\infty$, we have $\ln[(\beta-1)\bar{\zeta}(\theta)] \geq 0$ for θ close to 0. Hence

$$\liminf_{\theta\downarrow 0} \frac{\ln \mathbb{E}\left(\bar{Z}_\theta - \bar{\zeta}(\theta)\right)_+}{\bar{\zeta}^{\frac{\alpha}{\alpha-1}}(\theta)} \geq \liminf_{\theta\downarrow 0} \frac{\ln \mathbb{P}\left(\bar{Z}_\theta \geq \beta\bar{\zeta}(\theta)\right)}{\bar{\zeta}^{\frac{\alpha}{\alpha-1}}(\theta)} = -\beta^{\frac{\alpha}{\alpha-1}} J_\alpha(0),$$

where the last inequality follows from Lemma 6.8, applied with $\xi(\theta) = \beta\bar{\zeta}(\theta)$. Going back to (6.13), we deduce

$$\left(1 - \frac{1}{\alpha}\right) \liminf_{\theta\downarrow 0} \frac{\ln(\theta)}{\bar{\zeta}^{\frac{\alpha}{\alpha-1}}(\theta)} \geq -\beta^{\frac{\alpha}{\alpha-1}} J_\alpha(0).$$

Since β is arbitrary in $(1, +\infty)$, we have

$$\left(1 - \frac{1}{\alpha}\right) \limsup_{\theta\downarrow 0} \frac{|\ln(\theta)|}{\bar{\zeta}^{\frac{\alpha}{\alpha-1}}(\theta)} \leq J_\alpha(0).$$

Therefore

$$\liminf_{\theta\downarrow 0} \frac{\bar{\zeta}^{\frac{\alpha}{\alpha-1}}(\theta)}{|\ln(\theta)|} \geq \frac{\alpha-1}{\alpha J_\alpha(0)} = (\alpha\eta_0 I_0)^{\frac{1}{\alpha-1}},$$

which proves (6.12).

In order to derive an upper bound for $\limsup_{\theta\downarrow 0} \frac{\zeta(\theta)}{\theta^{1/\alpha}|\ln\theta|^{1-\frac{1}{\alpha}}}$, we first deduce from (6.3) a lower bound for $\mathbb{E}\left(e^{\bar{X}_\theta} - 1 - \zeta(\theta)\right)_+$. We have

$$\begin{aligned}\mathbb{E}\left(e^{(r-\delta)\theta + X_\theta} - 1 - \zeta(\theta)\right)_+ &= \mathbb{E}\left(e^{(r-\delta)\theta + X_\theta}\mathbb{1}_{\{X_\theta \geq \ln(1+\zeta(\theta)) - (r-\delta)\theta\}}\right) \\ &\quad -(1+\zeta(\theta))\mathbb{P}(X_\theta \geq \ln(1+\zeta(\theta)) - (r-\delta)\theta).\end{aligned}$$

Note that

$$\mathbb{P}(X_\theta \geq \ln(1+\zeta(\theta)) - (r-\delta)\theta) = \mathbb{P}\left(\frac{X_\theta}{\theta^{1/\alpha}} \geq \frac{\ln(1+\zeta(\theta)) - (r-\delta)\theta}{\theta^{1/\alpha}}\right).$$

Since $X_\theta/\theta^{1/\alpha}$ weakly converges to a finite random variable Z as $\theta \downarrow 0$ and $\lim_{\theta\downarrow 0}\zeta(\theta)/\theta^{1/\alpha} = +\infty$, we have

$$\lim_{\theta\downarrow 0} \mathbb{P}\left(\frac{X_\theta}{\theta^{1/\alpha}} \geq \frac{\ln(1+\zeta(\theta)) - (r-\delta)\theta}{\theta^{1/\alpha}}\right) = 0.$$

Note that we also have

$$\lim_{\theta\downarrow 0} \mathbb{E}\left(e^{(r-\delta)\theta + X_\theta} - 1 - \zeta(\theta)\right)_+ = 0.$$

Therefore

$$\lim_{\theta\downarrow 0}\mathbb{E}\left(e^{X_\theta}\mathbb{1}_{\{X_\theta\geq\ln(1+\zeta(\theta))-(r-\delta)\theta\}}\right)=0$$

and

$$\begin{aligned}\mathbb{E}\left(e^{(r-\delta)\theta+X_\theta}-1-\zeta(\theta)\right)_+ &= \mathbb{E}\left(e^{X_\theta}\mathbb{1}_{\{X_\theta\geq\ln(1+\zeta(\theta))-(r-\delta)\theta\}}\right)\\ &\quad -(1+\zeta(\theta))\mathbb{P}(X_\theta\geq\ln(1+\zeta(\theta))-(r-\delta)\theta)+o(\theta),\end{aligned}$$

so that, using (6.3),

$$\begin{aligned}(r-\delta)\theta &= \mathbb{E}\left(e^{X_\theta}-(1+\zeta(\theta))\right)\mathbb{1}_{\{X_\theta\geq\ln(1+\zeta(\theta))-(r-\delta)\theta\}}+o(\theta)\\ &\leq \mathbb{E}\left(e^{X_\theta}-1-\zeta(\theta)\right)_+ +o(\theta).\end{aligned}\tag{6.14}$$

Using the decomposition $X_\theta=\bar{X}_\theta+X^0_\theta-\theta\int(e^y-1)_+\nu(dy)$, the independence of $\bar{X}$ and X^0, and the equality $\mathbb{E}e^{\bar{X}_\theta}=1$ (which follows from Lemma 6.6), we have

$$\begin{aligned}\mathbb{E}\left(e^{X_\theta}-1-\zeta(\theta)\right)_+ &\leq \mathbb{E}\left(e^{X^0_\theta+\bar{X}_\theta}-1-\zeta(\theta)\right)_+\\ &= \mathbb{E}\left(\left(e^{X^0_\theta}-1\right)e^{\bar{X}_\theta}+e^{\bar{X}_\theta}-1-\zeta(\theta)\right)_+\\ &\leq \mathbb{E}\left(\left(e^{X^0_\theta}-1\right)e^{\bar{X}_\theta}\right)+\mathbb{E}\left(e^{\bar{X}_\theta}-1-\zeta(\theta)\right)_+\\ &= e^{\theta\int(e^y-1)_+\nu(dy)}-1+\mathbb{E}\left(e^{\bar{X}_\theta}-1-\zeta(\theta)\right)_+\\ &= \theta\int(e^y-1)_+\nu(dy)+\mathbb{E}\left(e^{\bar{X}_\theta}-1-\zeta(\theta)\right)_+ +o(\theta).\end{aligned}$$

Hence, going back to (6.14),

$$\left(r-\delta-\int(e^y-1)_+\nu(dy)\right)\theta\leq\mathbb{E}\left(e^{\bar{X}_\theta}-1-\zeta(\theta)\right)_+ +o(\theta).\tag{6.15}$$

Introducing the notation $l(\theta)=\ln(1+\zeta(\theta))$, we have

$$\begin{aligned}\mathbb{E}\left(e^{\bar{X}_\theta}-1-\zeta(\theta)\right)_+ &= \mathbb{E}\left(e^{\bar{X}_\theta}-e^{l(\theta)}\right)_+\\ &= \mathbb{E}\left(\mathbb{1}_{\{\bar{X}_\theta\geq l(\theta)\}}\int_{l(\theta)}^{\bar{X}_\theta}e^y dy\right)_+\\ &\leq \int_{l(\theta)}^{+\infty}e^y\mathbb{P}\left(\bar{X}_\theta\geq y\right)dy\\ &= \theta^{1/\alpha}\int_{\bar{l}(\theta)}^{+\infty}e^{z\theta^{1/\alpha}}\mathbb{P}\left(\bar{X}_\theta\geq z\theta^{1/\alpha}\right)dz,\end{aligned}$$

with $\bar{l}(\theta)=l(\theta)/\theta^{1/\alpha}$. It follows from Lemma 6.7 that, given any $a\in[a_0,0)$, we have

$$\mathbb{P}\left(\bar{X}_\theta\geq z\theta^{1/\alpha}\right)\leq\exp\left(C_a\theta^{1-\frac{1}{\alpha}}z^{\frac{1}{\alpha-1}}-J_\alpha(a)z^{\frac{\alpha}{\alpha-1}}\right).$$

Now, fix $\varepsilon > 0$. Since $\lim_{\theta\downarrow 0} \bar{l}(\theta) = +\infty$, we have, for θ close to 0,

$$\forall z \geq \bar{l}(\theta), \quad z\theta^{1/\alpha} + C_a \theta^{1-\frac{1}{\alpha}} z^{\frac{1}{\alpha-1}} \leq \varepsilon z^{\frac{\alpha}{\alpha-1}}.$$

Hence, with the notation $\hat{\alpha} = \frac{\alpha}{\alpha-1}$,

$$\mathbb{E}\left(e^{\bar{X}_\theta} - 1 - \zeta(\theta)\right)_+ \leq \theta^{1/\alpha} \int_{\bar{l}(\theta)}^{+\infty} e^{-(J_\alpha(a)-\varepsilon)z^{\hat{\alpha}}} dz.$$

We can assume ε close enough to 0 so that $J_\alpha(a) > \varepsilon$, and

$$\begin{aligned}\int_{\bar{l}(\theta)}^{+\infty} e^{-(J_\alpha(a)-\varepsilon)z^{\hat{\alpha}}} dz &\leq \frac{1}{\hat{\alpha}(J_\alpha(a)-\varepsilon)(\bar{l}(\theta))^{\hat{\alpha}-1}} \int_{\bar{l}(\theta)}^{+\infty} \hat{\alpha}(J_\alpha(a)-\varepsilon) z^{\hat{\alpha}-1} e^{-(J_\alpha(a)-\varepsilon)z^{\hat{\alpha}}} dz \\ &= \frac{1}{\hat{\alpha}(J_\alpha(a)-\varepsilon)(\bar{l}(\theta))^{\hat{\alpha}-1}} e^{-(J_\alpha(a)-\varepsilon)\bar{l}^{\hat{\alpha}}(\theta)}.\end{aligned}$$

Going back to (6.15), we deduce

$$\left(r - \delta - \int (e^y - 1)_+ \nu(dy)\right) \theta^{1-\frac{1}{\alpha}} \leq \frac{1}{\hat{\alpha}(J_\alpha(a)-\varepsilon)(\bar{l}(\theta))^{\hat{\alpha}-1}} e^{-(J_\alpha(a)-\varepsilon)\bar{l}^{\hat{\alpha}}(\theta)} + o(\theta^{1-\frac{1}{\alpha}}),$$

so that, for θ close to 0,

$$\theta^{1-\frac{1}{\alpha}} \leq \frac{C_\varepsilon}{(\bar{l}(\theta))^{\hat{\alpha}-1}} e^{-(J_\alpha(a)-\varepsilon)\bar{l}^{\hat{\alpha}}(\theta)},$$

where C_ε is a positive constant. Hence

$$\left(1 - \frac{1}{\alpha}\right) \ln\theta \leq -(J_\alpha(a)-\varepsilon)\bar{l}^{\hat{\alpha}}(\theta) + \ln\left(\frac{C_\varepsilon}{(\bar{l}(\theta))^{\hat{\alpha}-1}}\right),$$

and

$$\left(1 - \frac{1}{\alpha}\right) \limsup_{\theta\downarrow 0} \frac{\ln\theta}{\bar{l}^{\hat{\alpha}}(\theta)} \leq -(J_\alpha(a)-\varepsilon).$$

Since a and ε can be arbitrarily close to 0, we get, in the limit,

$$\limsup_{\theta\downarrow 0} \frac{\ln\theta}{\bar{l}^{\hat{\alpha}}(\theta)} \leq -\frac{\alpha}{\alpha-1} J_\alpha(0) = -(\alpha\eta_0 I_0)^{\frac{-1}{\alpha-1}}.$$

Note that $\lim_{\theta\downarrow 0} \frac{l(\theta)}{\zeta(\theta)} = 1$, so that we can conclude that

$$\limsup_{\theta\downarrow 0} \frac{\bar{\zeta}^{\hat{\alpha}}(\theta)}{|\ln\theta|} \leq (\alpha\eta_0 I_0)^{\frac{1}{\alpha-1}}.$$

◇

6.2 Estimating the difference $b_e - b$

Proposition 6.9 *Under assumption (AS), we have*

$$\limsup_{t\to T} \frac{b_e(t)-b(t)}{(T-t)^{1/\alpha}} < \infty.$$

Proof: It follows from the variational inequality and the inequality $\partial P/\partial t \le 0$ that, for $t \in (0,T)$ and $x \in (b(t),K)$, we have

$$(r-\delta)x\frac{\partial P}{\partial x}(t,x)+\int \left(P(t,xe^y)-P(t,x)-x\frac{\partial P}{\partial x}(t,x)(e^y-1)\right)\nu(dy) \ge r(K-x).$$

Note that, since $\int (e^y-1)_+\nu(dy)<\infty$, we may write

$$\int_{(0,+\infty)} \left(P(t,xe^y)-P(t,x)-x\frac{\partial P}{\partial x}(t,x)(e^y-1)\right)\nu(dy) =$$
$$\int_{(0,+\infty)} (P(t,xe^y)-P(t,x))\,\nu(dy) - x\frac{\partial P}{\partial x}(t,x)\int_{(0,+\infty)}(e^y-1)\,\nu(dy),$$

so that, using the notation

$$d^+ = r-\delta-\int (e^y-1)_+\,\nu(dy),$$

we get

$$\begin{aligned} &d^+x\frac{\partial P}{\partial x}(t,x)+\int_{(0,+\infty)}(P(t,xe^y)-P(t,x))\,\nu(dy)\\ &\quad+\int_{(-\infty,0)}\left(P(t,xe^y)-P(t,x)-x\frac{\partial P}{\partial x}(t,x)(e^y-1)\right)\nu(dy)\ge r(K-x).\end{aligned}$$

Therefore, for $x\in(b(t),K)$,

$$\int_{(-\infty,0)}\left(P(t,xe^y)-P(t,x)-x\frac{\partial P}{\partial x}(t,x)(e^y-1)\right)\nu(dy)\ge -d^+x\frac{\partial P}{\partial x}(t,x)-J(t),$$

where

$$J(t)=\int_{(0,+\infty)}\sup_{b(t)<x<K}|P(t,xe^y)-P(t,x)|\,\nu(dy).$$

Note that, due to the Lipschitz property of $P(t,.)$, we have, for $x\in(b(t),K)$ and $y>0$,

$$0\le P(t,x)-P(t,xe^y)\le x\,(e^y-1)\le K\,(e^y-1)\,,$$

and

$$P(t,x)-P(t,xe^y)\le P(t,b(t)).$$

Since $\lim_{t\to T} P(t,b(t)) = P(T,K) = 0$, we deduce $\lim_{t\to T} J(t) = 0$. Now, for $x \in (b(t), b_e(t))$, we have (with $\theta = T-t$)

$$\frac{\partial P}{\partial x}(t,x) \le \frac{\partial_- P}{\partial x}(t,b_e(t)) \le \frac{\partial_- P_e}{\partial x}(t,b_e(t)) = -\mathbb{E}\left(e^{-\delta\theta+X_\theta}\mathbb{1}_{\left\{b_e(t)e^{(r-\delta)\theta+X_\theta}\le K\right\}}\right),$$

where ∂_- refers to left-hand derivatives, the first inequality follows from the convexity of $P(,t,.)$ and the second inequality follows from the fact that $x \mapsto (P-P_e)(t,x)$ is non-increasing (see Corollary 3.4). Observe that

$$\begin{aligned}\mathbb{E}\left(e^{-\delta\theta+X_\theta}\mathbb{1}_{\left\{b_e(t)e^{(r-\delta)\theta+X_\theta}\le K\right\}}\right) &= \mathbb{E}\left(e^{-\delta\theta+X_\theta}\mathbb{1}_{\{(r-\delta)\theta+X_\theta\le\ln(1+\zeta(\theta))\}}\right)\\ &= \mathbb{E}\left(e^{-\delta\theta+X_\theta}\mathbb{1}_{\left\{\frac{(r-\delta)\theta+X_\theta}{\theta^{1/\alpha}}\le\frac{\ln(1+\zeta(\theta))}{\theta^{1/\alpha}}\right\}}\right).\end{aligned}$$

Using (6.4) and Lemma 6.3, we derive

$$\lim_{\theta\to 0}\mathbb{E}\left(e^{-\delta\theta+X_\theta}\mathbb{1}_{\left\{\frac{(r-\delta)\theta+X_\theta}{\theta^{1/\alpha}}\le\frac{\ln(1+\zeta(\theta))}{\theta^{1/\alpha}}\right\}}\right) = 1.$$

Now, for $x \in (b(t), K)$ denote

$$I(t,x) = \int_{(-\infty,0)}\left(P(t,xe^y) - P(t,x) - x\frac{\partial P}{\partial x}(t,x)(e^y-1)\right)\nu(dy).$$

It follows from the above discussion that

$$\liminf_{t\to T}\inf_{x\in(b(t),b_e(t))} I(t,x) \ge d^+K. \tag{6.16}$$

We will now derive an upper bound for $I(t,x)$, for $b(t) < x < K$. We have

$$\begin{aligned}I(t,x) = &\int_{\left(-\infty,\ln\frac{b(t)}{x}\right]}\left(P(t,xe^y) - P(t,x) - x\frac{\partial P}{\partial x}(t,x)(e^y-1)\right)\nu(dy)\\ &+\int_{\left(\ln\frac{b(t)}{x},0\right)}\left(P(t,xe^y) - P(t,x) - x\frac{\partial P}{\partial x}(t,x)(e^y-1)\right)\nu(dy).\end{aligned}$$

For $y \le \ln\frac{b(t)}{x}$, we have

$$\begin{aligned}P(t,xe^y) - P(t,x) - x\frac{\partial P}{\partial x}(t,x)(e^y-1) &= (K-xe^y) - P(t,x)\\ &\quad -x\frac{\partial P}{\partial x}(t,x)(e^y-1)\\ &\le (K-xe^y) - (K-x)\\ &\quad -x\frac{\partial P}{\partial x}(t,x)(e^y-1)\\ &= x\left(1+\frac{\partial P}{\partial x}(t,x)\right)(1-e^y).\end{aligned}$$

For $y \in (\ln(b(t)/x), 0)$, we have, using the convexity of $P(t, .)$,

$$\begin{aligned} P(t, xe^y) - P(t,x) - x\frac{\partial P}{\partial x}(t,x)(e^y - 1) &\le x(e^y-1)\frac{\partial P}{\partial x}(t, xe^y) - x\frac{\partial P}{\partial x}(t,x)(e^y-1) \\ &= x\left(\frac{\partial P}{\partial x}(t,x) - \frac{\partial P}{\partial x}(t,xe^y)\right)(1-e^y)\,. \end{aligned}$$

Therefore

$$\begin{aligned} I(t,x) \le x\left(1 + \frac{\partial P}{\partial x}(t,x)\right) & \int_{\left(-\infty, \ln\frac{b(t)}{x}\right]} (1-e^y)\,\nu(dy) \\ & + \int_{\left(\ln\frac{b(t)}{x}, 0\right)} x\left(\frac{\partial P}{\partial x}(t,x) - \frac{\partial P}{\partial x}(t,xe^y)\right)(1-e^y)\,\nu(dy). \quad (6.17) \end{aligned}$$

Due to (6.16), there exists $\eta > 0$ such that for $t \in [T-\eta, T)$,

$$\inf_{x \in (b(t), b_e(t))} I(t,x) \ge \frac{d^+K}{2}. \quad (6.18)$$

From now on, we assume $t \in [T-\eta, T)$ and, for $\xi \in (0, \ln(b_e(t)/b(t)))$ we set

$$g_t(\xi) = P(t, b(t)e^{\xi}).$$

Note that the derivative of g_t is given by

$$g_t'(\xi) = b(t)e^{\xi}\frac{\partial P}{\partial x}(t, b(t)e^{\xi}),$$

and, due to the smooth fit property, $g_t'(0) = -b(t)$. We also have $|g_t'(\xi)| \le b_e(t) \le K$. Applying (6.17) with $x = b(t)e^{\xi}$, we have, using (6.18),

$$\begin{aligned} \frac{d^+K}{2} \le & \left(g_t'(\xi) - g_t'(0)e^{\xi}\right) \int_{(-\infty,-\xi]} (1-e^y)\,\nu(dy) \\ & \int_{(-\xi,0)} \left(g_t'(\xi) - g_t'(\xi+y)e^{-y}\right)(1-e^y)\,\nu(dy) \end{aligned}$$

Note that

$$\begin{aligned} \left(g_t'(\xi) - g_t'(0)e^{\xi}\right) \int_{(-\infty,-\xi]} (1-e^y)\,\nu(dy) &= \left(g_t'(\xi) - g_t'(0)\right) \int_{(-\infty,-\xi]} (1-e^y)\,\nu(dy) \\ &\quad + g_t'(0)(1-e^{\xi}) \int_{(-\infty,-\xi]} (1-e^y)\,\nu(dy). \end{aligned}$$

For $\xi \in (0, \ln(b_e(t)/b(t)))$, we have, for any $\varepsilon > 0$,

$$\begin{aligned} \left(e^{\xi}-1\right)\int_{(-\infty,-\xi]} (1-e^y)\,\nu(dy) &\le \left(e^{\xi}-1\right)\nu\left((-\infty,-\varepsilon]\right) + \int_{(-\varepsilon,0)} \left(e^{-y}-1\right)(1-e^y)\,\nu(dy) \\ &\le \left(\frac{b_e(t)}{b(t)} - 1\right)\nu\left((-\infty,-\varepsilon]\right) + \int_{(-\varepsilon,0)} \left(e^{-y}-1\right)(1-e^y)\,\nu(dy). \end{aligned}$$

Therefore

$$\lim_{t\to T}\sup_{\xi\in(0,\ln(b_e(t)/b(t)))}\left(e^{\xi}-1\right)\int_{(-\infty,-\xi]}(1-e^y)\,\nu(dy)=0.$$

By taking η smaller if necessary, we can now assume that, for $t\in[T-\eta,T)$ and $\xi\in(0,\ln(b_e(t)/b(t)))$, we have

$$\begin{aligned}\frac{d^+K}{3}&\leq\left(g_t'(\xi)-g_t'(0)\right)\int_{(-\infty,-\xi]}(1-e^y)\,\nu(dy)\\&\quad+\int_{(-\xi,0)}\left(g_t'(\xi)-g_t'(\xi+y)e^{-y}\right)(1-e^y)\,\nu(dy).\end{aligned}\tag{6.19}$$

Now, take $a\in(0,\ln(b_e(t)/b(t)))$. By integrating (6.19) with respect to ξ from 0 to a, we get

$$\frac{d^+K}{3}a\leq j_1(a)+j_2(a),$$

where

$$j_1(a)=\int_0^a d\xi\,\left(g_t'(\xi)-g_t'(0)\right)\left(\int_{(-\infty,-\xi]}(1-e^y)\,\nu(dy)\right)$$

and

$$j_2(a)=\int_0^a d\xi\int_{(-\xi,0)}\nu(dy)\left(g_t'(\xi)-g_t'(\xi+y)e^{-y}\right)(1-e^y).$$

In order to estimate $j_1(a)$ we note that, for $\xi\in(0,a)$,

$$\begin{aligned}g_t'(\xi)&=b(t)e^{\xi}\frac{\partial P}{\partial x}(t,b(t)e^{\xi})\\&\leq b(t)e^{\xi}\frac{\partial P}{\partial x}(t,b(t)e^{a})=e^{\xi-a}g_t'(a)\leq e^{-a}g_t'(a),\end{aligned}$$

where the first inequality follows from the convexity of $P(t,.)$ and the second one from $g_t'(a)\leq 0$. Hence

$$\begin{aligned}j_1(a)&\leq\left(e^{-a}g_t'(a)-g_t'(0)\right)\int_0^a d\xi\left(\int_{(-\infty,-\xi]}(1-e^y)\,\nu(dy)\right)\\&=\left(e^{-a}g_t'(a)-g_t'(0)\right)\int_{-\infty}^0\nu(dy)\,(1-e^y)\left(\int_0^{a\wedge(-y)}d\xi\right).\end{aligned}$$

Note that $e^{-a}g_t'(a)-g_t'(0)=b(t)\left(1+\frac{\partial P}{\partial x}(t,b(t)e^a)\right)\geq 0$. Using the assumptions we have on ν, we can find $\beta<0$ such that, for $y\in(\beta,0)$,

$$\nu(dy)\leq\frac{2\eta_0}{|y|^{1+\alpha}}dy,$$

so that, using $e^y \geq 1+y$,

$$
\begin{aligned}
j_1(a) &\leq \left(e^{-a} g_t'(a) - g_t'(0)\right) \left(a \int_{(-\infty,\beta]} \nu(dy)\,(1-e^y) + \int_{(\beta,0)} \frac{2\eta_0 dy}{|y|^{1+\alpha}}\,(1-e^y) \left(\int_0^{(a\wedge -y)} d\xi \right)\right) \\
&\leq \left(e^{-a} g_t'(a) - g_t'(0)\right) \left(a\nu((-\infty,\beta]) + \int_0^a d\xi \int_\xi^{|\beta|} \frac{2\eta_0 dy}{|y|^{1+\alpha}} |y| \right) \\
&\leq \left(e^{-a} g_t'(a) - g_t'(0)\right) \left(a\nu((-\infty,\beta]) + \frac{2\eta_0}{\alpha-1} \int_0^a \xi^{1-\alpha} d\xi \right) \\
&= \left(e^{-a} g_t'(a) - g_t'(0)\right) \left(a\nu((-\infty,\beta]) + \frac{2\eta_0}{(2-\alpha)(\alpha-1)} a^{2-\alpha} \right)
\end{aligned}
$$

Note that $a \in (0, \ln(b_e(t)/b(t)))$ and $\lim_{t\to T} \ln(b_e(t)/b(t)) = 0$. So, for t close enough to T, we may assume $a \in (0,1]$, so that $a \leq a^{2-\alpha}$ (recall $1<\alpha<2$). Therefore, for some $C>0$,

$$
\begin{aligned}
j_1(a) &\leq C a^{2-\alpha} \left(e^{-a} g_t'(a) - g_t'(0)\right) \\
&= C a^{2-\alpha} \left(g_t'(a) - g_t'(0)\right) + C g_t'(a) a^{2-\alpha} \left(e^{-a} - 1\right) \\
&\leq C a^{2-\alpha} \left(g_t'(a) - g_t'(0)\right) + C K a^{3-\alpha}. \qquad (6.20)
\end{aligned}
$$

We now study $j_2(a)$. Note that, for $y<0$,

$$
g_t'(\xi) - g_t'(\xi+y) e^{-y} = b(t) e^{\xi} \left(\frac{\partial P}{\partial x}\left(t, b(t)e^{\xi}\right) - \frac{\partial P}{\partial x}\left(t, b(t) e^{\xi+y}\right)\right) \geq 0.
$$

Since $a \in (0, \ln(b_e(t)/b(t)))$ and $\lim_{t\to T} \ln(b_e(t)/b(t)) = 0$, we may assume $a < |\beta|$ and write

$$
\begin{aligned}
j_2(a) &\leq \int_0^a d\xi \int_{(-\xi,0)} \frac{2\eta_0}{|y|^{1+\alpha}} dy \left(g_t'(\xi) - g_t'(\xi+y)e^{-y}\right)(1-e^y) \\
&= 2\eta_0 \int_0^a d\xi \int_0^\xi \frac{dy}{y^{1+\alpha}} \left(g_t'(\xi) - g_t'(\xi-y)e^y\right)\left(1-e^{-y}\right) \\
&\leq 2\eta_0 \int_0^a d\xi \int_0^\xi \frac{dy}{y^\alpha} \left(g_t'(\xi) - g_t'(\xi-y)e^y\right),
\end{aligned}
$$

where the last inequality follows from $1-e^{-y} \leq y$. Hence

$$
\begin{aligned}
j_2(a) &\leq 2\eta_0 \int_0^a \frac{dy}{y^\alpha} \int_y^a d\xi \left(g_t'(\xi) - g_t'(\xi-y)e^y\right) \\
&= 2\eta_0 \int_0^a \frac{dy}{y^\alpha} \left(g_t(a) - g_t(a-y)e^y - g_t(y) + g_t(0)e^y\right). \\
&= 2\eta_0 \int_0^a \frac{dy}{y^\alpha} \left(g_t(a) - g_t(a-y) - g_t(y) + g_t(0)\right) + 2\eta_0 \int_0^a \frac{dy}{y^\alpha}(e^y-1)\left(g_t(0) - g_t(a-y)\right) \\
&\leq 2\eta_0 \int_0^a \frac{dy}{y^\alpha} \left(g_t(a) - g_t(a-y) - g_t(y) + g_t(0)\right) + 2\eta_0 K a \int_0^a \frac{dy}{y^\alpha}(e^y-1),
\end{aligned}
$$

where the last inequality follows from $||g_t'||_\infty \leq K$. Note that $a\int_0^a \frac{dy}{y^\alpha}(e^y-1) \leq Ca^{3-\alpha}$ for some $C>0$. On the other hand, we have, for $y \in (0,a)$,

$$\begin{aligned}
g_t(a) - g_t(a-y) &= \int_0^y g_t'(a-z)dz \\
&= \int_0^y b(t)e^{a-z}\frac{\partial P}{\partial x}\left(t, b(t)e^{a-z}\right)dz \\
&\leq \int_0^y b(t)e^{a-z}\frac{\partial P}{\partial x}\left(t, b(t)e^{a}\right)dz \\
&\leq \int_0^y b(t)\frac{\partial P}{\partial x}\left(t, b(t)e^{a}\right)dz = yb(t)\frac{\partial P}{\partial x}\left(t, b(t)e^{a}\right) = ye^{-a}g_t'(a),
\end{aligned}$$

where the first inequality follows from the convexity of $P(t,.)$ and the second one from $\partial P/\partial x \leq 0$. Similarly, we have

$$\begin{aligned}
g_t(y) - g_t(0) = \int_0^y g_t'(z)dz &= \int_0^y b(t)e^{z}\frac{\partial P}{\partial x}\left(t, b(t)e^{z}\right)dz \\
&\geq \int_0^y b(t)e^{z}\frac{\partial P}{\partial x}\left(t, b(t)\right)dz \\
&\geq yb(t)e^{y}\frac{\partial P}{\partial x}\left(t, b(t)\right) = g_t'(0)ye^y.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^a \frac{dy}{y^\alpha}\left(g_t(a) - g_t(a-y) - g_t(y) + g_t(0)\right) &\leq \int_0^a \frac{dy}{y^\alpha} y\left(e^{-a}g_t'(a) - e^y g_t'(0)\right) \\
&= \int_0^a \frac{dy}{y^{\alpha-1}}\left(g_t'(a) - g_t'(0)\right) \\
&\quad + \int_0^a \frac{dy}{y^{\alpha-1}}\left[\left(e^{-a}-1\right)g_t'(a) + \left(1-e^y\right)g_t'(0)\right] \\
&\leq \int_0^a \frac{dy}{y^{\alpha-1}}\left(g_t'(a) - g_t'(0)\right) \\
&\quad +(a + (e^a - 1))K\frac{a^{2-\alpha}}{2-\alpha} \\
&\leq Ca^{2-\alpha}\left(g_t'(a) - g_t'(0)\right) + Ca^{3-\alpha},
\end{aligned}$$

for some $C>0$, so that we have

$$j_2(a) \leq Ca^{2-\alpha}\left(g_t'(a) - g_t'(0)\right) + Ca^{3-\alpha} \tag{6.21}$$

Putting (6.20) and (6.21) together, we conclude that, for some positive constant C, we have

$$\frac{d^+K}{3}a \leq Ca^{2-\alpha}\left(g_t'(a) - g_t'(0)\right) + Ca^{3-\alpha}$$

or, equivalently,

$$\frac{d^+K}{3C}a^{\alpha-1}\left(1-\frac{3C}{d^+K}a^{2-\alpha}\right)\leq g_t'(a)-g_t'(0).$$

For t close enough to T, we have, for all $a\in(0,\ln(b_e(t)/b(t)))$, $a^{2-\alpha}<\frac{d^+K}{6C}$, hence

$$\frac{d^+K}{6C}a^{\alpha-1}\leq g_t'(a)-g_t'(0).$$

We now integrate this inequality with respect to a from 0 to $a_t=\ln(b_e(t)/b(t))$ to derive

$$a_t^\alpha\leq C\left(g_t(a_t)-a_tg_t'(0)-g_t(0)\right),$$

where C is a positive constant. Hence

$$\begin{aligned}\frac{1}{C}(b_e(t)-b(t))^\alpha &\leq P(t,b_e(t))+b(t)\ln\frac{b_e(t)}{b(t)}-P(t,b(t))\\ &\leq P(t,b_e(t))+b_e(t)-b(t)-(K-b(t))\\ &= P(t,b_e(t))-P_e(t,b_e(t))\leq rK(T-t),\end{aligned}$$

where the last inequality follows from the Early Exercise Premium Formula. ◇

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